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Violation of the phase space general covariance as a diffeomorphism anomaly in quantum mechanics

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Abstract

We consider a topological quantum mechanics described by a phase space path integral and study the 1-dimensional analog for the path integral representation of the Kontsevich formula. We see that the naive bosonic integral possesses divergences, that it is even naively non-invariant and thus is ill-defined. We then consider a super-extension of the theory which eliminates the divergences and makes the theory naively invariant. This super-extension is equivalent to the correct choice of measure and was discussed in the literature. We then investigate the behavior of this extended theory under diffeomorphisms of the extended phase space and despite of its naive invariance find out that the theory possesses anomaly under nonlinear diffeomorphisms. We localize the origin of the anomaly and calculate the lowest nontrivial anomalous contribution.

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1 Introduction

The notion of classical symmetry is often changed when proceeding to the quantum world. Depending on the way one fixes the quantum theory, the symmetry can be preserved, deformed or broken. In the path integral approach the symmetries are treated in the most classical-like way and the freedom of quantization consists in the freedom of choice of regularization, while in an operator approach this freedom consists in the choice of ordering (in both approaches quantum corrections are possible as well).

When the symmetry can not be preserved one gets a quantum anomaly - a phenomenon widely known in the context of quantum field theories. Although this is usually connected with field theories in the even-dimensional space-time, in the case of 1-dimensional QFT - quantum mechanics - one still can find interesting effects that can be treated as anomalies.

This is closely related to another possibility for the classical symmetries in quantum world - possibility to be deformed. In such case the classical symmetry seems to be broken and this can be treated as the anomaly, however symmetry can be restored by adding quantum corrections and thus replacing the notion of classical symmetry by some quantum analog.

In the quantum mechanical case we know that the symmetries of the classical Poisson manifold are deformed at the quantum level in general. Namely, in the framework of deformation quantization one may be interested in the symmetries of a particular star-product. The classical symmetries form the subgroup inside classical diffeomorphisms of the Poisson manifold and the classical diffeomorphisms do not preserve the star-product. Instead they change the star-product into the gauge-equivalent one (see the corresponding paper by Kontsevich [1]).

The notion of diffeomorphism can be deformed via quantum corrections in order to make the star-product covariant under such modified transformations (see paper [2] where an appropriate modification is described and discussed in terms of L_∞-morphism of Kontsevich). This may indicate that the theory contains anomalies (as understood in the usual sense - the theory is non-invariant under classical transformations).

We are going to investigate this in a current paper. The theory under consideration is a topological quantum mechanics with a classical limiting Hamiltonian system on a Poisson manifold (Hamiltonian is taken to be 0 – that is the meaning of "topological"). We are going to analyze the covariance of this system. Covariance of the system can be thought of as a special kind of symmetry which means that the objects of the theory behave in a good geometric way like in the classical case (e.g. like tensors). If we had the covariance preserved then there would exist a covariant formula for the star-product in terms of the Poisson structure $\theta^{ij}(x)$ (i.e. the form of this formula in terms of $\theta^{ij}(x)$ would be invariant). However such formula doesn't exist, as we have stated before, which indicates the breakdown of covariance.

We also may think in a following way. Kontsevich's deformation quantization is formulated in terms of the 2-dimensional QFT with quantum-mechanical observables living on the boundary (see [3] for the path integral formulation). However it would be instructive to reformulate the quantization procedure in terms of the 1-dimensional QFT, quantum me-

chanics by itself. And one could expect that we have such an approach - a naively covariant path integral. Indeed, we have the simplest star-product - a famous Moyal product (which works for the case of the constant Poisson structure $\theta^{ij} = const$):

$$(f \star_M g)(x) = f(x) e^{\frac{i\hbar}{2} \overleftarrow{\partial_i} \theta^{ij} \overrightarrow{\partial_j}} g(x) \tag{1}$$

Now consider a mechanical system with phase space $\mathbb{R}^d$ (where d is even), with time taking values on a circle $t \in S^1$, $t \in [-\pi, \pi]$ and with the Poisson structure $\theta^{ij} = const$. A well-known fact (see e.g. [3] or [4]) is that the Moyal product can be written in terms of the path integral as a special correlator of the topological quantum mechanics, namely as follows:

$$(f \star_M g)(x) = \frac{\int \mathcal{D}\phi \prod_{i=1}^d \delta(\phi^i(\pm\pi) - x^i) f(\phi(t_1))\, g(\phi(t_2))\, e^{\frac{i}{\hbar}\int \frac{1}{2}\omega_{ij}\phi^i \dot{\phi}^j dt}}{\int \mathcal{D}\phi \prod_{i=1}^d \delta(\phi^i(\pm\pi) - x^i) e^{\frac{i}{\hbar}\int \frac{1}{2}\omega_{ij}\phi^i \dot{\phi}^j dt}} \tag{2}$$

where ω is the symplectic structure, i.e. it is inverse to the Poisson structure: $\omega_{ij}\theta^{jk} = \delta_i^k$. But this gives us a way to make a straightforward but naive generalization of the Moyal product to the case of the non-constant[1] θ^{ij}:

$$(f \star g)(x) = \frac{\int \mathcal{D}\phi \prod_{i=1}^d \delta(\phi^i(\pm\pi) - x^i) f(\phi(t_1))\, g(\phi(t_2))\, e^{\frac{i}{\hbar}\int \alpha}}{\int \mathcal{D}\phi \prod_{i=1}^d \delta(\phi^i(\pm\pi) - x^i) e^{\frac{i}{\hbar}\int \alpha}} \tag{3}$$

where α is a 1-form such that $d\alpha = \omega$. From the naive point of view this formula could give us a covariant way to define the star-product for the case of an arbitrary $\theta^{ij}(x)$. And naively it could be invariant under diffeomorphisms. But we have discussed that there is no way to write the star-product on the symplectic (or Poisson) manifold covariant under *classical* diffeomorphisms without introduction of some additional structures.

This indicates again that (3) is non-invariant in fact and contains some kind of anomaly responsible for this non-invariance.

The goal of the current paper is to built the corresponding framework and to study the anomaly in the simplest case where it shows itself. We are not going to deal with the whole expression (3) yet, neither to check if it really is a star-product (i.e. if it is associative) - that will be the topic of the upcoming research.

[1] In fact one can provide a naive reasoning for this formula. In [3] the following path integral representation for the Kontsevich quantization formula had been obtained:

$$(f \star g)(x) = \int_{X(\infty)=x} f(X(1)) g(X(0)) e^{\frac{i}{\hbar} S[X,p]}$$

with fields X and p living on a disk D^2, three points $0, 1, \infty$ fixed on a boundary of the disk and with action

$$S[X,p] = \int_{D^2} p_i \wedge dX^i + \frac{1}{2}\theta^{ij} p_i \wedge p_j$$

If one formally integrates over p's in this formula then all the bulk-dependence naively drops out and we are left with a boundary theory:

$$(f \star g)(x) = \int_{X(\infty)=x} \widetilde{\mathcal{D}X} f(X(1)) g(X(0)) e^{\frac{i}{\hbar}\int \alpha}$$

with $d\alpha = \omega$ and with a special naive measure $\widetilde{\mathcal{D}X} = \prod_{t \in \partial D^2} \sqrt{\det \omega\big(X(t)\big)} d^d X(t)$. Although this is not exactly the formula (3), but rather the formula to be discussed in the "Super improvement" section, it is instructive to start with a "wrong" formula (3) to see why it fails and only then study the naively correct one.

In the chapter "Path integral for the phase space" it will be shown that the object (3) is an ill-defined object in fact. A supersymmetric improvement of it will be provided as well as the regularization which is always crucial for the path integral. After such improvements (3) will become well-defined and finite.

We'll provide some illustrative calculations in the chapter "Examples", namely we'll prove the formula (2), check the classical limit of (3) and show peculiarities of loop calculations in our theory.

The chapter "Anomaly chasing" is devoted to the understanding of how the diffeomorphism acts in our theory. We'll discuss how to perform diffeomorpism compatible with the regularization describing the two approaches for this - the naive one and the proper one. We'll obtain the anomaly in the lowest order of perturbation theory in the naive approach and then we'll explain its nature in the proper approach.

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2 Path integral for the phase space

2.1 Naive bosonic approach

2.1.1 Formulation

In order to build a framework we should define a path integral $\int \mathcal{D}\phi e^{\frac{i}{\hbar}\int \alpha}$ at first, where $\phi^i(t)$ is a map $\phi : S^1 \to M$ where M is the d-dimensional symplectic manifold (d is even) with symplectic form $\omega = d\alpha$ and S^1 is the time manifold.

We'll provide a standard perturbation theory later and thus decompose the action into the quadratic and non-quadratic parts. But such decomposition makes sense only when $M = \mathbb{R}^d$ - the property of being quadratic is purely coordinate and has no global meaning in the case of an arbitrary manifold. So we need to consider $M = \mathbb{R}^d$ as soon as we need a perturbation theory. There's also another reason for such a restriction related with measure.

We have to define measure $\mathcal{D}\phi$ in some way. There are basically two ways to do that - we can either provide time-slicing (i.e. lattice regularization) or provide mode decomposition. We are not going into details here - we just mention that the first procedure is ill-defined and needs some extra analysis in the case of the phase space. So we choose the second one according to which we have to provide Fourier decomposition of the fields:

$$\phi^i(t) = \sum_{n=-\infty}^{\infty} \phi^i_n e^{int} \tag{4}$$

where $\phi^i_n \in \mathbb{C}$ and $\phi^i_{-n} = \overline{\phi^i_n}$ (the requirement of reality of $\phi^i(t)$) and then construct the measure in an appropriate way:

$$\mathcal{D}\phi \sim \prod_n \prod_i d\phi^i_n = \prod_i d\phi^i_0 \prod_{n>0} \prod_i d\phi^i_n d\overline{\phi^i_n} \tag{5}$$

regardless if it is well-defined yet.

Possibility to make this is the second evidence for the manifold M to be just $\mathbb{R}^d$. Indeed, Fourier decomposition (4) makes no sense until we can take linear combinations of the fields, which means that M should be a linear space.

Next we put:

$$\alpha = \frac{1}{2}\omega_{ij}^{(0)}\phi^i d\phi^j + e_i(\phi)d\phi^i \tag{6}$$

$$\omega = d\alpha = \omega_{ij}^{(0)}d\phi^i \wedge d\phi^j + de \tag{7}$$

where $\omega_{ij}^{(0)}$ is constant and provide perturbation theory expansions in $e_i(\phi)$.

2.1.2 Non-invariance of the measure

At this point we can already notice that measure (5) is by no means invariant under diffeomorphisms of the symplectic manifold. Indeed, if we provide a coordinate change, the Jacobian will appear and the general covariance will be broken: $\mathcal{D}\phi \sim \left|\frac{\delta\phi}{\delta\phi'}\right|\mathcal{D}\phi'$ where $\left|\frac{\delta\phi}{\delta\phi'}\right|$ is a non-constant functional and thus it contributes into the correlators. So the answer obtained within such a prescription is not expected to be covariant.

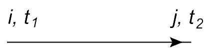
2.1.3 Study of ultraviolet divergences

However, non-invariance is not the only pathology of the described object - the case is that it is divergent as we'll see later and even non-renormalizable in a standard QFT sense.

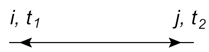
Indeed, introduce diagram notations for the propagators of the non-interacting theory (arrow notation for the derivative is similar to the fat dot notation from the book [5]):

$${}^0\left\langle\left(\phi^i(t_1)-x^i\right)\left(\phi^j(t_2)-x^j\right)\right\rangle =$$

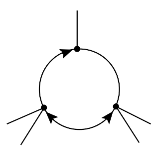
$${}^0\left\langle\left(\phi^i(t_1)-x^i\right)\dot{\phi}^j(t_2)\right\rangle =$$



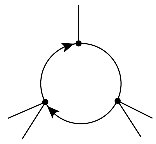
$${}^0\left\langle\dot{\phi}^i(t_1)\dot{\phi}^j(t_2)\right\rangle =$$



where notation ${}^0\langle...\rangle$ means that correlator under consideration is the free one, i.e. of the quadratic theory. As soon as the action is of the first-order type $S_0 = \int dt\frac{1}{2}\omega_{ij}\phi^i\dot{\phi}^j$, we can conclude that in the momentum space the first propagator is proportional to the inverse momentum $\frac{1}{p}$, the second one is ~ 1 and the third one is $\sim p$. This means that every wheel-like diagram with equal number of internal derivatives (denoted by arrows) and internal propagators:



is linearly divergent. Diagrams of the type:



where the number of internal propagators exceeds the number of internal arrows by 1, are logarithmical divergent, but here we can notice that in the case of an odd-dimensional world-sheet (which is our case - it is 1-dimensional in our theory) logarithmical divergences do not occur due to the symmetries of the integrand. Indeed, expression of the type $\int_{a<|p|<\Lambda} \frac{d^n p}{p^n}$ is $\propto \log \Lambda$ for even n and equals to 0 for odd n. In the case of odd n the result depends on the form of the cut-off and thus the divergence is replaced by the ambiguity. But existence of an infinite series of the linear divergent wheel-like diagrams indicates that the theory is not just divergent but even non-renormalizable. One can find more detailed discussion in the Appendix A.

So what do we see? Theory that looks very natural at the first sight happened to be extremely ill-defined. Although it was 1-dimensional it happened to be non-renormalizable due to the interactions containing derivatives. Fortunately all the troubles can be solved in a one simple step - that is the topic of the following subsection.

2.2 Super improvement

We can restore measure invariance and theory finiteness by introducing a generally-covariant measure:

$$\mathcal{D}\phi \sim \prod_t \sqrt{\det \omega\left(\phi(t)\right)} d\phi^1(t)...d\phi^d(t) \tag{8}$$

where d is even. A piece of luck is that all the needed counterterms to cancel the divergences are present in this choice of measure. To see that we'll lift the square root of $\det \omega$ by introducing an anticommuting real *ghost* field ψ^i (in the spirit of [6] and [5] where the ghosts are intensively used in the configuration space theory):

$$\sqrt{\det \omega\left(\phi(t)\right)} \sim \int d\psi^1(t)...d\psi^d(t) e^{\frac{i}{\hbar}\omega_{ij}(\phi(t))\psi^i(t)\psi^j(t)} \tag{9}$$

and thus consider a modified super measure $\mathcal{D}\phi\mathcal{D}\psi$.

Such measure has already been talked about in the literature in different contexts. Connection of covariant measure (8) and supersymmetric measure $\mathcal{D}\phi\mathcal{D}\psi$ was discussed in ([7]) and ([8]) and essentially used in ([9]).

One can notice without further considerations that an obtained extended theory with an action $S = \int dt \left\{\alpha_i(\phi)\dot{\phi}^i + \omega_{ij}(\phi)\psi^i\psi^j\right\}$ possesses supersymmetry, i.e. it is invariant under:

$$\delta\phi^i = \theta\psi^i, \qquad \delta\psi^i = -\theta\dot{\phi}^i$$

with θ being small anticommuting parameter (see ([9]) again). However, we'll not need this property in a further consideration at all.

The main advantages of the super improvement are manifest measure invariance and divergences cancelation. In this subsection we are going to define a quantum mechanical path integral based on such a modified approach and check its consistency.

2.2.1 The construction

Suppose we have a symplectic manifold (M, ω) where M is a d-dimensional (d is even) smooth manifold and ω is a symplectic form. We have already discussed before that there should be a linear structure on M in order to define a perturbative path integral with mode-regularized measure, so we consider $M = \mathbb{R}^d$. We also have a time manifold S^1. Let us parameterize it by $t \in [-\pi, \pi]$. Consider a continuous and thus integrable map:

$$\phi : S^1 \to \mathbb{R}^d \tag{10}$$

This map is just a d-component real boson field on S^1. We describe it in terms of its Fourier transform:

$$\phi^k(t) = \sum_{n\in\mathbb{Z}} \phi^k_n e^{int} \qquad (k = 1..d) \tag{11}$$

where ϕ^i_n are complex bosons with

$$\phi^i_{-n} = \overline{\phi^i_n} \tag{12}$$

We also consider a d-component fermion field $\psi^i(t),\ i = 1..d$. We define it as a set of d time-dependent linear combinations in the infinite-dimensional Grassmann algebra:

$$\psi^k(t) = \sum_{n\in\mathbb{Z}} \psi^k_n e^{int} \qquad (k = 1..d) \tag{13}$$

where ψ^i_n are fermion Fourier modes - the complex Grassmann variables (except of ψ^i_0 which is real) satisfying:

$$\psi^i_{-n} = \overline{\psi^i_n}. \tag{14}$$

Here we should notice that due to the reality condition (12) the zeroth boson mode ϕ^i_0 is real while $(\phi^i_1, \phi^i_{-1} = \overline{\phi^i_1})$, $(\phi^i_2, \phi^i_{-2} = \overline{\phi^i_2})$, etc. each takes values in $\mathbb{C}$. But as integration over fermions has a formal algebraic meaning (as opposed to the boson integrals that have analytical sense) we can think of them due to (14) in two equivalent ways: either say that ψ^i_0 is real and $(\psi^i_1, \psi^i_{-1} = \overline{\psi^i_1})$, $(\psi^i_2, \psi^i_{-2} = \overline{\psi^i_2})$ etc. are complex, i.e. each form $\Pi\mathbb{C}$ or say that all the fermions ..., $\psi^i_{-2},\ \psi^i_{-1},\ \psi^i_0,\ \psi^i_1,\ \psi^i_2, ...$ are real. We will prefer the first understanding due to its analogy to the bosonic case.

We define the mode-regularized fields as follows:

$$\phi^i_N(t) = \sum_{n=-N}^{N} \phi^i_n e^{int}$$

$$\psi^i_N(t) = \sum_{n=-N}^{N} \psi^i_n e^{int} \tag{15}$$

Notice that regularized bosons take values in $\mathbb{R}^d \times \mathbb{C}^{Nd}$, where $\phi_0^i \in \mathbb{R}$ and $(\phi_k^i, \phi_{-k}^i) \in \mathbb{C}$. Fix the action functional:

$$S[\phi, \psi] = \int \left\{ \alpha_i(\phi)\, \dot{\phi}^i + \omega_{ij}(\phi)\, \psi^i \psi^j - H(\phi) \right\} dt \tag{16}$$

where $\omega = d\alpha$ and take a functional $F[\phi, \psi]$.

Definition 1. Mode-regularized functional integral in the phase space is an expression:

$$I_N[F, S] = \mathcal{A}(N) \int \prod_{n=-N}^{N} \left[d\phi_n^1 ... d\phi_n^d d\psi_n^1 ... d\psi_n^d \right] F[\phi_N, \psi_N] e^{\frac{i}{\hbar} S[\phi_N, \psi_N]} \tag{17}$$

where integration over ψ's is understood in the sense of Berezin, integration over ϕ's is provided throughout all the space $\mathbb{R}^d \times \mathbb{C}^{Nd}$ and $\mathcal{A}(N)$ does not depend on $F[\phi, \psi]$.

We'll say that functional integral exists in the sense of mode-regularization if one can find such a function $\mathcal{A}(N)$ (that does not depend on $F[\phi, \psi]$) that $\lim_{N\to\infty} I_N$ exists.

In practice we will understand (17) in a perturbative way, namely we'll expand the non-quadratic part of the action into the Taylor series, integrate each term and understand existence of the path integral as existence of all the terms in the resulting formal series. When we choose $\mathcal{A}(N) = \frac{1}{I_N[1,S]}$ we get absolute correlators (averaged quantities):

Definition 2.

1) An absolute regularized correlator of $F[\phi, \psi]$ is the following:

$$\frac{I_N[F, S]}{I_N[1, S]} \tag{18}$$

2) A relative regularized correlator of $F[\phi, \psi]$ for a given functional $G[\phi, \psi]$ is the following:

$$\frac{I_N[FG, S]}{I_N[G, S]} \tag{19}$$

The appropriate limits $N \to \infty$ (if they do exist) are called absolute and relative correlators. Notice that if the absolute correlator exists so does the relative one.

In our main case of interest the Hamiltonian will be equal to zero $H(\phi) = 0$ and we'll consider the following relative correlators:

$$\langle F[\phi, \psi] \rangle_N = \frac{I_N[F\eta_x[\phi], S]}{I_N[\eta_x[\phi], S]} \tag{20}$$

$$\langle F[\phi, \psi] \rangle = \lim_{N\to\infty} \frac{I_N[F\eta_x[\phi], S]}{I_N[\eta_x[\phi], S]} \tag{21}$$

where $\eta_x[\phi] = \prod_i \delta\left(\phi^i(\pi) - x^i\right)$ and we have introduced a widespread notation $\langle ... \rangle$ for the correlator.

Notice that in the classical limit $\eta_x[\phi]$ is nothing but an "evaluation observable" - it evaluates the value of the inserted observable $F[\phi]$ at the classical solution with $\phi^i(\pi) = x^i$ (if such solution exists) - in our case such solution really exists and it is just $\phi^i(t) = x^i = const$. If the inserted observable is the product of functions: $F[\phi] = f_1(\phi(t_1))\, f_2(\phi(t_2))\, ... f_n(\phi(t_n))$ then in the limit $\hbar \to 0$ (21) gives rise to an ordinary point-wise product of functions evaluated at the point x: $f_1(x) f_2(x) ... f_n(x)$. This will be proved in the section 3.2 where we'll provide the tree level calculation of such correlator.

From technical point of view $\eta_x[\phi]$ is an infrared regulator (zero mode regulator) - we need it due to the action $\int_{S^1} \frac{1}{2}\omega_{ij}^{(0)}\phi^i\dot{\phi}^j dt$ being invariant under global translations $\phi^i \to \phi^i + c^i$.

Then we extract the quadratic part of the action in order to provide perturbations. To do that we use (6)-(7) and get $S = S_0 + S_{int}$ where:

$$S_0 = \int \left\{ \frac{1}{2}\omega_{ij}^{(0)}\phi^i\dot{\phi}^j + \omega_{ij}^{(0)}\psi^i\psi^j \right\} dt \tag{22}$$

$$S_{int} = \int \left\{ e_i(\phi)\dot{\phi}^i + de_{ij}\psi^i\psi^j \right\} dt \tag{23}$$

Notice that the choose of background (22) is not coordinate-invariant, however it is preserved by the linear transformations as well as the linear structure on M.

Free correlators

Consider free correlators, i.e. correlators of the theory with action S_0. Introduce appropriate notations ${}^0\langle ... \rangle_N$, ${}^0\langle ... \rangle$ (such notation have already been used in 2.1.3). We calculate them explicitly:

$$^0\langle \phi_N^i(t_1)\phi_N^j(t_2)\rangle_N = x^i x^j + i\hbar\theta_{(0)}^{ij} G_N(t_1,t_2) \tag{24}$$

Where $\theta_{(0)} = \left(\omega^{(0)}\right)^{-1}$, $G_N(t_1,t_2) = \sum_{n,m=-N}^{N} G_{n,m}e^{int_1+imt_2}$ and:

$$G_{n,m} = \frac{1}{2\pi i}\left\{ \frac{\delta_{n+m}}{n} - \frac{\delta_m}{n}(-1)^n + \frac{\delta_n}{m}(-1)^m \right\}$$

$$G_{0,0} = 0 \tag{25}$$

The answer for ψ's can be calculated in the same way and is given by:

$$^0\langle \psi_n^i \psi_n^j \rangle_N = \frac{i\hbar}{2}\theta_{(0)}^{ij}\frac{\delta_{n+m}}{2\pi} \tag{26}$$

or equivalently:

$$^0\langle \psi_N^i(t_1)\psi_N^j(t_2)\rangle_N = \frac{i\hbar}{2}\theta_{(0)}^{ij}\delta_N(t_1-t_2) \tag{27}$$

where $\delta_N(t)$ is a regularized delta-function:

$$\delta_N(t) = \frac{1}{2\pi}\sum_{n=-N}^{N} e^{int} \tag{28}$$

In continuous limit:

$$^0\langle \psi^i(t_1)\psi^j(t_2)\rangle = \frac{i\hbar}{2}\theta_{(0)}^{ij}\delta(t_1-t_2) \tag{29}$$

And in (24):

$$G(t_1,t_2) = \begin{cases} \frac{1}{2}Sign(t_1-t_2), & if\ \ -\pi < t_1, t_2 < \pi; \\ 0, & if\ t_1 = \pi \text{ or } t_2 = \pi. \end{cases} \tag{30}$$

We should also notice that $G_N(t_1,t_2) = -G_N(t_2,t_1)$ and:

$$\frac{\partial G_N(t_1,t_2)}{\partial t_1} = \delta_N(t_1-t_2) - \delta_N(t_1-\pi) \tag{31}$$

Now we can use perturbation theory in order to compute correlation functions up to the fixed order in couplings.

It is important to note that when calculating perturbations we take self-contractions into account. Indeed, from one point of view there is no reason to neglect them. From another point of view if we ignored such terms we could possibly break covariance under diffeomorphisms and thus it would need additional analysis which we don't want to provide for the purposes of simplicity.
We'll use the Feynman diagrams technics described above for convenience.

2.2.2 Naive expectations

The first pathology that is claimed to be got rid of by the supersymmetric improvement is measure non-invariance. Let us observe how it works from the naive point of view. From the expression (16) we see that fermions should transform like vector fields under diffeomorphisms (in order to make action a scalar), namely if we provide:

$$\phi^i = \phi^i(\varphi) \tag{32}$$

then we should make a substitution in a fermion sector (here $\widetilde{\psi}$ is a fermion):

$$\psi^i = \widetilde{\psi}^j \frac{\partial \phi^i}{\partial \varphi^j} \tag{33}$$

But now we can observe that

$$d^d\phi(t) d^d\psi(t) = \left[\det \frac{\partial \phi^i(\varphi(t))}{\partial \varphi^j(t)}\right] d^d\varphi \left[\det \frac{\partial \phi^i(\varphi(t))}{\partial \varphi^j(t)}\right]^{-1} d^d\widetilde{\psi} = d^d\varphi d^d\widetilde{\psi} \tag{34}$$

and thus naively conclude that in the functional case product of the standard boson measure and the Berezin fermion measure $\mathcal{D}\phi\mathcal{D}\psi$"="$\prod_t d^d\phi(t)d^d\psi(t)$ is invariant under diffeomorphisms. To get rid of the word "naively" one should reexamine it in a regularization (as the path integral is meaningless without regularization). We'll return to this question later when discussing the diffeomorphisms and regularization - we will find out that the measure is really invariant with some additional assumptions about diffeomorphism (to be discussed in the Anomaly chasing chapter).

We've seen that the starting bosonic theory was divergent. Now we claim that the supersymmetric improvement solves this problem - the theory becomes finite.
We can provide a loose argument of finiteness here. Since the theory is naively invariant under diffeomorphisms, we can find such coordinates p_i, q^i (as it is stated by the Darboux theorem), that $\omega = \sum_{i=1}^{d/2} dp_i \wedge dq^i$ and thus we can choose $\alpha = \sum_{i=1}^{d/2} p_i dq^i$ to make the action quadratic. But quadratic theory is pretty well-defined - we don't need any perturbations in it. So does the starting theory.

The only possible divergences can arise if we consider some special divergent observables, e.g. ${}^0\langle \phi^i(t)\dot{\phi}^j(t)\rangle$ is divergent even in the quadratic theory. To get rid of such cases we'll consider only observables of the type $f_1(\phi(t_1))f_2(\phi(t_2))...f_n(\phi(t_n))$ further.

Our naive argument is straightforward but wrong as long as we don't know weather the mode-regularized theory is still invariant with respect to diffeomorphisms. Moreover, it will be shown that the invariance is broken but it's breakdown is finite and in some sense the argument above is not as bad. However to provide a successive theory we should find a

rigorous argumentation at this point. We will provide a detailed perturbative analysis to prove the finiteness.

2.2.3 Cancelation of divergences: loop analysis

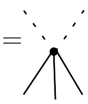
Consider correlation function of the monomial:

$$\langle \phi^{i_1}(t_1)...\phi^{i_k}(t_k)\rangle \tag{35}$$

We'll prove that (35) is finite and piecewise-continuous (i.e. it can contain jumps but not infinities).
In order to do that we should describe diagram technics. It is almost the same as before except of existence of anticommuting ghosts. So we introduce notation for the free ghost propagator:

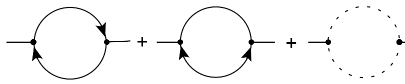
$$^0\langle \psi^i(t_1)\psi^j(t_2)\rangle =$$ 

And introduce a new type of vertexes corresponding to the couplings of ψ's with ϕ's:

$$\frac{i}{\hbar}\int dt\ \omega_{ij}(\phi)\psi^i\psi^j =$$ 

where the number of external bosons is not fixed as in general case $e_i(\phi)\dot{\phi}^i$ interaction contains all the possible terms of the type $\frac{1}{k!}\partial_{j_1}...\partial_{j_k}e_i(0)\phi^{j_1}...\phi^{j_k}\dot{\phi}^i$. Now let us make several observations.
Observation 1: Ghosts can occur only in loops. This follows immediately from that ghosts contribute quadratically to the appropriate vertexes. Thus we can neglect ghosts on a tree level. Moreover, every ghost loop is a circle with exterior bosonic legs.
Observation 2: Propagator of bosons (25) doesn't preserve momentum while vertexes as long as the propagator of ghosts (26) preserve it. This is natural because we've broken translation invariance on a world line when regularizing 0-mode by the δ-function.
Observation 3: All tree-level diagrams are finite. This statement holds because the tree level reproduces the classical answer which is definitely finite - the argumentation is standard here. At the tree level one can safely take a limit $N \to \infty$ and work with integrals of distributions - everything happens to be well-defined then.
So we have to check possible loop diagrams and prove that their sum is finite.
Observation 4: There are no logarithmic divergences in theories with an odd-dimensional world-sheet - instead we have logarithmic ambiguities. We have already mentioned this in the section 2.1.3.
So now all we need to do is to show that loops do not give rise to the linear (or higher) divergences. There exist three types of loops in our theory: ghost-loops, non-ghost-loops and loops containing both ghost and non-ghost fields. But as soon as ghosts always form loops, it is enough to show cancelation of such loops - then we'll get rid of ghosts at all. Finiteness of correlators will follow from this immediately.

A step-by-step loop analysis can be found in the Appendix B. Here we only state that the main effect is that both divergent bosonic loops and ghost loops contain $\delta_N(0)$ divergences, that are in fact linear divergences as $\delta_N(0) \sim \sum_{-N}^{N} 1$. Then we find that to every ghost loop there corresponds a set of divergent bosonic loops with the same number of external legs. We evaluate them for the finite N and find that divergent parts (i.e. expressions proportional to N) cancel out and we finally get what we needed. We can take a limit $N \to \infty$ safely at the end. The diagram illustration is as follows:



So we have shown that our super-improved approach gives rise to the finite theory. However in the Appendix B this is discussed only in the case of some special system - quantum mechanics on a circle with zero hamiltonian and with delta-function fixing fields at $t = \pi$. This can be generalized to an arbitrary non-zero hamiltonian in a straightforward way. We'll not show this explicitly but only give an idea.

Divergencies will always arise from the non-ghost loops with equal number of internal propagators and derivatives (internal arrows). The point is that propagator of the type $\langle \phi^i(t)\phi^j(0)\rangle$ always contains jump at $t = 0$ as soon as $\lim_{t\to+0}\left[\langle \phi^i(t)\phi^j(0)\rangle - \langle \phi^j(t)\phi^i(0)\rangle\right] = i\hbar\langle\theta^{ij}\rangle + O(\hbar^2)$ due to the usual quantum mechanical correspondence principle. This jump gives rise to the δ-function after differentiation which causes the $\delta_N(0)$-divergency after all. And such divergencies are exactly canceled by the ghost loops. Finally we get a finite one-dimensional quantum field theory as expected.

Notice that from the "Naive expectations" subsection we know that finiteness follows from the invariance. We have shown that the theory is finite. Then it still can be invariant or non-invariant. We'll check which case is true in the "Anomaly chasing" section.

3 Examples

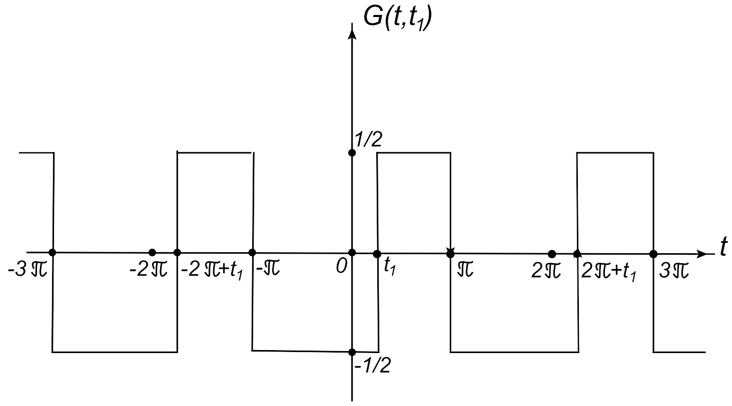
3.1 Free correlation functions

3.1.1 Propagators

The free propagators are described by the formulas (24)-(30). Notice that $G_{n,m}$ has such a form that

$$\sum_{n,m=-N}^{N} G_{n,m} e^{i(n+m)t} = 0 \qquad (36)$$

and thus in our prescription we have $G(t,t) = G_N(t,t) = 0$. As we have already mentioned, $G(t,\pi) = G(\pi,t) = 0$ and so the propagator has the structure as follows:



As this function is not smooth and we often deal with operations that demand smoothness (or at least differentiability), we should always be careful and work with regularized expression (with cut-off parameter N) which is smooth and take $N \to \infty$ limit only at the end.

3.1.2 Regularized delta-function

In a section 2 we have introduced a regularized delta-function $\delta_N(t) = \frac{1}{2\pi}\sum_{n=-N}^{N} e^{int}$. In the current subsection we are going to collect some of its elementary properties as long as they will be widely used below.

1) According to (28) the function $\delta_N(t)$ is normalized in such a way, that

$$\int \delta_N(t)dt = 1 \tag{37}$$

2) $\delta_N(t)$ acts like a real delta-function on the space of functions, which Fourier transforms have the support lying inside the set $\{-N, -N+1, ..., N-1, N\}$, i.e. it acts like a real delta-function on the space of Fourier polynomials $span\{e^{-iNt}, e^{-i(N-1)t}, ..., e^{iNt}\}$. Indeed, if we have $P(t) = \sum_{k=-N}^{N} p_k e^{ikt}$, then:

$$\begin{aligned}\int P(t)\delta_N(t-t_0)dt &= \textstyle\sum_{n,k=-N}^{N} p_k \frac{1}{2\pi}\int e^{i(k+n)t-int_0}dt \\ &= \textstyle\sum_{n,k=-N}^{N} p_k\delta_{k+n}e^{-int_0} = \sum_{k=-N}^{N} p_k e^{ikt_0} = P(t_0)\end{aligned} \tag{38}$$

3) $\delta_N(t)$, integrated with an arbitrary function, projects it on the $span\{e^{-iNt}, e^{-i(N-1)t}, ..., e^{iNt}\}$ subspace and acts like a real delta-function there. Indeed, consider the function $f(t) = \sum_{k=-\infty}^{\infty} f_k e^{ikt}$. Then:

$$\begin{aligned}\int f(t)\delta_N(t-t_0)dt &= \textstyle\sum_{n=-N}^{N} \frac{1}{2\pi}\int f(t)e^{int-int_0}dt \\ &= \textstyle\sum_{n=-N}^{N} f_{-n}e^{-int_0} = \sum_{n=-N}^{N} f_n e^{int_0}\end{aligned} \tag{39}$$

Where the last expression is nothing but the function $f(t)$ which higher modes have been cut-off. It is convenient to introduce the following notation for this projection:

Notation:

$$[f]_N(t) = \sum_{k=-N}^{N} f_k e^{ikt} = \int f(t')\delta_N(t'-t)dt' \tag{40}$$

3.1.3 Moyal product revisited

Now we want to show that the correlator ${}^0\langle f_1(\phi(t_1))f_2(\phi(t_2))\rangle$ gives rise to the famous Moyal product as mentioned in the introduction. But first we should note that ${}^0\langle\phi^i(t)\rangle = x^i$ and as ${}^0\langle(\phi^i(t)-x^i)\rangle = 0$:

Observation: Wick theorem is held only for monomials over the variables $\phi^i(t)-x^i$, e.g.:

$$\begin{aligned}
&{}^0\langle(\phi^{i_1}(t_1)-x^{i_1})(\phi^{i_2}(t_2)-x^{i_2})(\phi^{i_3}(t_3)-x^{i_3})(\phi^{i_4}(t_4)-x^{i_4})\rangle \\
= &\ {}^0\langle(\phi^{i_1}(t_1)-x^{i_1})(\phi^{i_2}(t_2)-x^{i_2})\rangle\ {}^0\langle(\phi^{i_3}(t_3)-x^{i_3})(\phi^{i_4}(t_4)-x^{i_4})\rangle + \\
&+ {}^0\langle(\phi^{i_1}(t_1)-x^{i_1})(\phi^{i_3}(t_3)-x^{i_3})\rangle\ {}^0\langle(\phi^{i_2}(t_2)-x^{i_2})(\phi^{i_4}(t_4)-x^{i_4})\rangle
\end{aligned} \tag{41}$$

As soon as $G(t,t)=0$ and ${}^0\langle(\phi^i(t_1)-x^i)(\phi^j(t_2)-x^j)\rangle = i\hbar\theta^{ij}_{(0)}G(t_1,t_2)$, we have ${}^0\langle(\phi^i(t)-x^i)(\phi^j(t)-x^j)\rangle = 0$ and thus it is straightforward to show that ${}^0\langle f(\phi(t))\rangle = f(x)$. After this comment the derivation of the Moyal product becomes trivial - we provide a Taylor expansion around $\phi^i = x^i$:

$$\begin{aligned}
{}^0\langle f_1(\phi(t_1))f_2(\phi(t_2))\rangle &= f_1(x)f_2(x) + \partial_i f_1(x)\partial_j f_2(x)\ {}^0\left\langle(\phi^i(t_1)-x^i)(\phi^j(t_2)-x^j)\right\rangle + \\
+\frac{1}{(2!)^2}\partial_{i_1}\partial_{i_2}f_1(x)\partial_{j_1}\partial_{j_2}f_2(x)\ &{}^0\left\langle(\phi^{i_1}(t_1)-x^{i_1})(\phi^{i_2}(t_1)-x^{i_2})(\phi^{j_1}(t_2)-x^{j_1})(\phi^{j_2}(t_2)-x^{j_2})\right\rangle + \ldots \\
&= f_1(x)f_2(x) + \partial_i f_1(x)\partial_j f_2(x)\ {}^0\left\langle(\phi^i(t_1)-x^i)(\phi^j(t_2)-x^j)\right\rangle + \\
+\frac{1}{2!}\partial_{i_1}\partial_{i_2}f_1(x)\partial_{j_1}\partial_{j_2}f_2(x)\ &{}^0\left\langle(\phi^{i_1}(t_1)-x^{i_1})(\phi^{j_1}(t_2)-x^{j_1})\right\rangle\ {}^0\left\langle(\phi^{i_2}(t_1)-x^{i_2})(\phi^{j_2}(t_2)-x^{j_2})\right\rangle + \ldots \\
&= f_1(x)f_2(x) + \partial_i f_1(x)\partial_j f_2(x) i\hbar\theta^{ij}_{(0)}G(t_1,t_2) + \\
&+\frac{1}{2!}\partial_{i_1}\partial_{i_2}f_1(x)\partial_{j_1}\partial_{j_2}f_2(x) i\hbar\theta^{i_1j_1}_{(0)}G(t_1,t_2) i\hbar\theta^{i_2j_2}_{(0)}G(t_1,t_2) + \ldots \\
&= f_1(x)\exp\left(i\hbar\overleftarrow{\partial_i}\theta^{ij}_{(0)}\overrightarrow{\partial_j}G(t_1,t_2)\right)f_2(x)
\end{aligned} \tag{42}$$

If $t_1 > t_2$ then

$$ {}^0\langle f_1(\phi(t_1))f_2(\phi(t_2))\rangle = f_1(x)\exp\left(\frac{i\hbar}{2}\overleftarrow{\partial_i}\theta^{ij}_{(0)}\overrightarrow{\partial_j}\right)f_2(x) \tag{43}$$

which is just the Moyal product.

3.2 Perturbation: the first order in $\hbar$

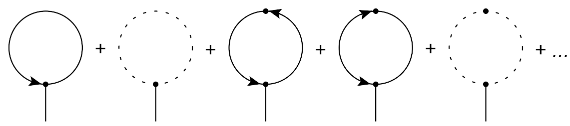
It is well-known that the tree level of the perturbation theory is of the first order in $\hbar$ and reproduces the Poisson structure. However there exist loop diagrams that could contribute to the $O(\hbar)$ part of correlators and could spoil this property. These loop contributions would definitely fall out of the commutators, as the commutators *have to* give the Poisson structure in the first order in $\hbar$. In this subsection we check that such loop contributions vanish in the $O(\hbar)$ and that in the first order in $\hbar$ correlators give rise exactly to the Poisson structure. To figure out this we act as follows. Consider:

$$\frac{{}^0\left\langle f_1(\phi_N(t_1))f_2(\phi_N(t_2))\exp\left(\frac{i}{\hbar}\int\left\{e_i(\phi_N)\dot\phi^i_N + (de)_{ij}\psi^i_N\psi^j_N\right\}dt\right)\right\rangle_N}{{}^0\left\langle\exp\left(\frac{i}{\hbar}\int\left\{e_i(\phi_N)\dot\phi^i_N + (de)_{ij}\psi^i_N\psi^j_N\right\}dt\right)\right\rangle_N} \tag{44}$$

Observation: We have the following:

$$\frac{{}^{0}\left\langle\left(\phi^i_N(t)-x^i\right)\exp\left(\frac{i}{\hbar}\int\left\{e_i(\phi_N)\dot{\phi}^i_N+(de)_{ij}\psi^i_N\psi^j_N\right\}dt\right)\right\rangle_N}{{}^{0}\left\langle\exp\left(\frac{i}{\hbar}\int\left\{e_i(\phi_N)\dot{\phi}^i_N+(de)_{ij}\psi^i_N\psi^j_N\right\}dt\right)\right\rangle_N}=O(\hbar^2) \tag{45}$$

The absence of the $O(1)$ term is obvious - the only such contribution is ${}^{0}\left\langle\left(\phi^i_N(t)-x^i\right)\right\rangle_N=0$. $O(\hbar)$ contribution arises from the following graphs:



(46)

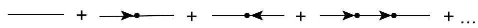
The appropriate analytical expression of the n^{th} order contribution is proportional to:

$$\begin{aligned}&\theta^{ik}_{(0)}\theta^{j_1i_2}_{(0)}\theta^{j_2i_3}_{(0)}\ldots\theta^{j_ni_1}_{(0)}\partial_k(de)_{i_1j_1}(de)_{i_2j_2}\ldots(de)_{i_nj_n}\times\\ &\times\int dt_1\ldots dt_n\Big[\Big(\delta_N(t_1-t_2)-\delta_N(t_2-\pi)\Big)\Big(\delta_N(t_2-t_3)-\delta_N(t_3-\pi)\Big)\ldots\times\\ &\times\Big(\delta_N(t_n-t_1)-\delta_N(t_1-\pi)\Big)-\delta_N(t_1-t_2)\delta_N(t_2-t_3)\ldots\delta_N(t_n-t_1)\Big]G_N(t,t_1)\end{aligned} \tag{47}$$

and vanishes due to $G_N(t,\pi)=0$ (as soon as $G_N(t,t')$ as a function of t has the support $\{-N,-N+1,...,N-1,N\}$ in the Fourier space, we can take into account the second property of $\delta_N(t)$ and work with $\delta_N(t)$ as with the standart delta-function in (47)). Now consider:

$$\frac{{}^{0}\left\langle\left(\phi^i_N(t_1)-x^i\right)\left(\phi^j_N(t_2)-x^j\right)\exp\left(\frac{i}{\hbar}\int\left\{e_i(\phi_N)\dot{\phi}^i_N+(de)_{ij}\psi^i_N\psi^j_N\right\}dt\right)\right\rangle_N}{{}^{0}\left\langle\exp\left(\frac{i}{\hbar}\int\left\{e_i(\phi_N)\dot{\phi}^i_N+(de)_{ij}\psi^i_N\psi^j_N\right\}dt\right)\right\rangle_N} \tag{48}$$

The $O(1)$ term is absent again, while the $O(\hbar)$ term is given by the sum over the tree diagrams:



We should notice that every pair gives rise to the $de(x)$ factor after integrating by parts. So the appropriate analytical expression is:

$$\begin{aligned}&i\hbar\theta^{ij}_{(0)}G(t_1,t_2)+i^3\hbar\theta^{ii_1}_{(0)}\big(\partial_{i_1}e_{j_1}(x)-\partial_{j_1}e_{i_1}(x)\big)\theta^{j_1j}_{(0)}\int G(t_1,t)\frac{dG(t,t_2)}{dt}dt+\\ &+i^5\hbar\theta^{ii_1}_{(0)}(de)_{i_1j_1}\theta^{j_1i_2}_{(0)}(de)_{i_2j_2}\theta^{j_2j}_{(0)}\int G(t_1,t)\frac{dG(t,t')}{dt}\frac{dG(t',t_2)}{dt'}dtdt'+\ldots\\ &=i\hbar G(t_1,t_2)\left(\theta_{(0)}-\theta_{(0)}de\theta_{(0)}+\theta_{(0)}de\theta_{(0)}de\theta_{(0)}-...\right)^{ij}=i\hbar G(t_1,t_2)\theta^{ij}\end{aligned} \tag{49}$$

where $\theta=\left(\omega^{(0)}+de\right)^{-1}$ and we omitted the obvious index notations inside the last brackets. We used (31) here. Notice that we took off the regularization in (49), i.e. put $N=\infty$ - it is acceptable at the tree level as it is well known in QFT. Taking into account (49) and that $G_N(t,t)=0$ we can conclude that (48) vanishes in the case $t_1=t_2=t$.

One more thing to mention is that higher degree monomials in $\phi^i(t)-x^i$ (higher than 2) giver rise to the $O(\hbar^2)$ terms and thus can be neglected.

Now it's easy to see that:

$$\frac{{}^{0}\left\langle f_1(\phi(t_1))f_2(\phi(t_2))\exp\left(\frac{i}{\hbar}\int\left\{e_i(\phi)\dot{\phi}^i+(de)_{ij}\psi^i\psi^j\right\}dt\right)\right\rangle}{{}^{0}\left\langle\exp\left(\frac{i}{\hbar}\int\left\{e_i(\phi)\dot{\phi}^i+(de)_{ij}\psi^i\psi^j\right\}dt\right)\right\rangle}=f_1(x)f_2(x)+$$

$$+\partial_i f_1(x)\partial_j f_2(x)\frac{{}^0\left\langle \left(\phi^i(t_1)-x^i\right)\left(\phi^j(t_2)-x^j\right)\exp\left(\frac{i}{\hbar}\int\left\{e_i(\phi)\dot{\phi}^i+(de)_{ij}\psi^i\psi^j\right\}dt\right)\right\rangle}{{}^0\left\langle \exp\left(\frac{i}{\hbar}\int\left\{e_i(\phi)\dot{\phi}^i+(de)_{ij}\psi^i\psi^j\right\}dt\right)\right\rangle}+$$
$$+O(\hbar^2)=f_1(x)f_2(x)+\frac{i\hbar}{2}\theta^{ij}\partial_i f_1(x)\partial_j f_2(x)+O(\hbar^2) \tag{50}$$

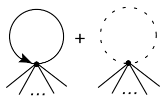
where we have assumed that $t_1>t_2$. As predicted, the $O(\hbar)$ term of the correlator (the tree level term) is nothing but the Poisson structure.

3.3 Loop calculations

In this subsection we will compute the simplest loop diagram to demonstrate some special features of the subject.

3.3.1 Anomalous vertex: first order in e

In the first order in perturbation theory loops can be obtained only by self-contractions. As long as ${}^0\langle\left(\phi^i_N(t)-x^i\right)\left(\dot{\phi^j_N}(t)-x^j\right)\rangle_N = 0$, only ${}^0\langle\left(\phi^i_N(t)-x^i\right)\dot{\phi}^j_N(t)\rangle_N$ contraction contributes. Therefore we are left with the following diagrams:



We will think of this as of a part of the bigger diagram. The analytical expression is:

$$\frac{1}{n!}\frac{i}{\hbar}\int dt\left(\phi^{i_1}_N(t)-x^{i_1}\right)\ldots\left(\phi^{i_n}_N(t)-x^{i_n}\right)\Big(\partial_{i_1}\ldots\partial_{i_n}\partial_i e_j(x)\ {}^0\langle\phi^i_N(t)\dot{\phi}^j_N(t)\rangle_N+$$
$$+\partial_{i_1}\ldots\partial_{i_n}\left(de(x)\right)_{ij}\ {}^0\langle\psi^i_N(t)\psi^j_N(t)\rangle_N\Big) \tag{51}$$

${}^0\langle\phi^i\dot{\phi}^j\rangle$ is proportional to θ^{ij} and therefor is antisymmetric in i,j. So we can rewrite (51) as follows:

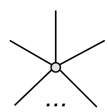
$$\frac{1}{n!}\frac{i}{\hbar}\int dt\left(\phi^{i_1}_N(t)-x^{i_1}\right)\ldots\left(\phi^{i_n}_N(t)-x^{i_n}\right)\partial_{i_1}\ldots\partial_{i_n}\left(de(x)\right)_{ij}\Big[\frac{1}{2}\ {}^0\langle\phi^i_N(t)\dot{\phi}^j_N(t)\rangle_N+$$
$$+\ {}^0\langle\psi^i_N(t)\psi^j_N(t)\rangle_N\Big] \tag{52}$$

The expression inside the square brackets is:

$$\begin{aligned}\tfrac{1}{2}\ {}^0\langle\phi^i_N(t)\dot{\phi}^j_N(t)\rangle_N+\ {}^0\langle\psi^i_N(t)\psi^j_N(t)\rangle_N &= \frac{i\hbar}{2}\theta^{ij}_{(0)}\left[\delta_N(t-\pi)-\delta_N(0)+\delta_N(0)\right]\\ &= \frac{i\hbar}{2}\theta^{ij}_{(0)}\delta_N(t-\pi)\end{aligned} \tag{53}$$

where $\delta_N(t)$ is a mode-regularized δ-function (28) and we have used (31). Substituting (53) into (52) we get an effective n-boson vertex:

$$V_n=-\frac{1}{2}\frac{1}{n!}\int dt\left(\phi^{i_1}_N(t)-x^{i_1}\right)\ldots\left(\phi^{i_n}_N(t)-x^{i_n}\right)\partial_{i_1}\ldots\partial_{i_n}\left(de(x)\right)_{ij}\theta^{ij}_{(0)}\delta_N(t-\pi)\ \ = \tag{54}$$



we can rewrite it using our projector:

$$V_n = -\frac{1}{2}\frac{1}{n!}\partial_{i_1}...\partial_{i_n}\left(de(x)\right)_{ij}\theta^{ij}_{(0)}\left[\left(\phi^{i_1}_N - x^{i_1}\right)...\left(\phi^{i_n}_N - x^{i_n}\right)\right]_N(\pi) \tag{55}$$

and then notice that:

$$V = \sum_{n=0}^{\infty} V_n = -\frac{1}{2}\theta^{ij}_{(0)}\left[de_{ij}\right]_N(\pi) \tag{56}$$

which is the claimed anomalous vertex as we'll see later. The contribution depends on the number of legs n.

Single leg case

In a case $n = 1$ (55) gives rise to the following:

$$V_1 = -\frac{1}{2}\partial_{i_1}\left(de(x)\right)_{ij}\theta^{ij}_{(0)}\left[\phi^{i_1}_N - x^{i_1}\right]_N(\pi) = -\frac{1}{2}\partial_{i_1}\left(de(x)\right)_{ij}\theta^{ij}_{(0)}\left(\phi^{i_1}_N(\pi) - x^{i_1}\right) = 0 \tag{57}$$

Here we used:

$$\left[\phi^i_N\right]_N(t) = \phi^i_N(t) \tag{58}$$

The last expression in (57) vanishes due to the delta-function fixing $\phi^i_N(\pi) = x^i$.

We could also contract this single leg to the external field $\phi^a(t_1)$, then the left-hand side of (57) would be proportional to

$$\int G_N(t_1,t)\delta_N(t-\pi)dt = G_N(t_1,\pi) = 0 \tag{59}$$

Notice that although expression $\int G(t_1,t)\delta(t-\pi)dt$ is ill-defined ($G(t_1,t)$ is discontinuous at $t=\pi$), we still can make sense of it and say $\int G_N(t_1,t)\delta_N(t-\pi)dt = G_N(t_1,\pi) = 0$ in a mode-regularization sense - we've already used this in the section 3.2 when explaining that the diagrams (46) didn't contribute. The situation will be pretty different with two or more legs - the delta-function $\delta_N(t)$ will not contain enough modes to make naive identities like $\int G_N(t_1,t)G_N(t_2,t)\delta_N(t-t')dt = G_N(t_1,t')G_N(t_2,t')$ correct.

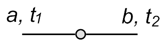
Two and more legs case

Consider

$$V_2 = -\frac{1}{4}\int dt\left(\phi^{i_1}_N(t) - x^{i_1}\right)\left(\phi^{i_2}_N(t) - x^{i_2}\right)\partial_{i_1}\partial_{i_2}\left(de(x)\right)_{ij}\theta^{ij}_{(0)}\delta_N(t-\pi) \tag{60}$$

which is no longer zero because in the Fourier space the support of $\left(\phi^{i_1}_N(t) - x^{i_1}\right)\left(\phi^{i_2}_N(t) - x^{i_2}\right)$ is $\{-2N,...,2N\}$.

To see this in a different manner let us contract the two exterior legs from (60) with exterior fields $\phi^a_N(t_1)$ and $\phi^b_N(t_2)$:



This gives rise to:

$$-\frac{(i\hbar)^2}{2}\theta^{ai_1}_{(0)}\theta^{bi_2}_{(0)}\theta^{ij}_{(0)}\partial_{i_1}\partial_{i_2}\left(de(x)\right)_{ij}\int G_N(t_a,t)G_N(t_b,t)\delta_N(t-\pi)dt \tag{61}$$

Introduce a notation[2]:

$$A^{(2)}(t_a,t_b)=\lim_{N\to\infty}A_N^{(2)}(t_a,t_b)=\lim_{N\to\infty}\int G_N(t_a,t)G_N(t_b,t)\delta_N(t-\pi)dt \tag{62}$$

The statement that (60) really contributes means that $A^{(2)}\neq 0$ and we finally get:

$$^0\langle\phi_N^a(t_1)V_2\phi_N^b(t_2)\rangle=\frac{\hbar^2}{2}\theta_{(0)}^{ai_1}\theta_{(0)}^{bi_2}\theta_{(0)}^{ij}\partial_{i_1}\partial_{i_2}\left(de(x)\right)_{ij}A^{(2)}(t_a,t_b) \tag{63}$$

This answer will be crucial for us as it will describe the lowest anomaly in our theory.

3.3.2 Nontrivial integral

We introduced the function $A^{(2)}(t_1,t_2)$ above. Understanding of it's behavior is important to understand the anomaly that will be discussed in the last section of the paper.

Let us demonstrate some special issues of this function. One could naively think according to (62) that the limit can be safely taken at the very beginning and that $A^{(2)}(t_a,t_b)=\int G(t_a,t)G(t_b,t)\delta(t-\pi)dt$, but this expression is ill-defined as $G(t',t)$ contains a jump at $t=\pi$. To be more consistent let us consider the following:

$$A_{N_1,N_2}^{(2)}(t_a,t_b)=\int G_{N_1}(t_a,t)G_{N_1}(t_b,t)\delta_{N_2}(t-\pi)dt \tag{64}$$

and observe how this behaves when $N_1,N_2\to\infty$. From one point:

$$\begin{aligned}\lim_{\substack{N_1\to\infty\\ \frac{N_2}{N_1}\to\infty}}A_{N_1,N_2}^{(2)}(t_a,t_b)&=\lim_{N_1\to\infty}\int G_{N_1}(t_a,t)G_{N_1}(t_b,t)\delta(t-\pi)dt\\ &=\lim_{N_1\to\infty}G_{N_1}(t_a,\pi)G_{N_1}(t_b,\pi)=0\end{aligned} \tag{65}$$

From another point of view we could do the following:

$$\lim_{\substack{N_2\to\infty\\ \frac{N_1}{N_2}\to\infty}}A_{N_1,N_2}^{(2)}(t_a,t_b)=\lim_{N_2\to\infty}\int G(t_a,t)G(t_b,t)\delta_{N_2}(t-\pi)dt \tag{66}$$

But we know the behavior of $G(t',t)$ in the vicinity of the point $t=\pi$ - it is constant and equal to $-\frac{1}{2}$ to the left, constant and equal to $\frac{1}{2}$ to the right and vanishes at $t=\pi$. Thus $G(t_1,t)G(t_2,t)$ is constant and equal to $\frac{1}{4}$ at the vicinity of $t=\pi$ except of the point $t=\pi$ itself where it vanishes. But integration with a smooth function $\delta_{N_2}(t-\pi)$ is insensitive to this single discontinuity and thus we conclude that:

$$\lim_{\substack{N_2\to\infty\\ \frac{N_1}{N_2}\to\infty}}A_{N_1,N_2}^{(2)}(t_a,t_b)=\frac{1}{4} \tag{67}$$

So what have we got? The answer depends on how we take the limit and thus we cannot put $N=\infty$ at the very beginning. We should compute (62) and take $N\to\infty$ at the end.

[2] In the n-legs case we'll obtain the similar integral:

$$A^{(n)}(t_1,t_2,...,t_n)=\lim_{N\to\infty}\int G_N(t_1,t)G_N(t_2,t)...G_N(t_n,t)\delta_N(t-\pi)dt$$

Let us do this in details. Substituting Fourier transforms of propagators and delta-function into (62) one gets:

$$
\begin{aligned}
A_N^{(2)}(t_1,t_2) &= \sum_{n_1,m_1,n_2,m_2,p=-N}^{N} \frac{1}{2\pi}\int G_{n_1m_1}e^{in_1t_1+im_1t}G_{n_2m_2}e^{in_2t_2+im_2t}e^{ip(t-\pi)}dt \\
&= \sum_{n_1,m_1,n_2,m_2,p=-N}^{N} e^{in_1t_1+in_2t_2}G_{n_1m_1}G_{n_2m_2}(-1)^p\delta_{m_1+m_2+p} \\
&= \sum_{n_1,m_1,n_2,m_2=-N}^{N} e^{in_1t_1+in_2t_2}G_{n_1m_1}G_{n_2m_2}(-1)^{m_1+m_2}\delta(|m_1+m_2|\le N)
\end{aligned}
\tag{68}
$$

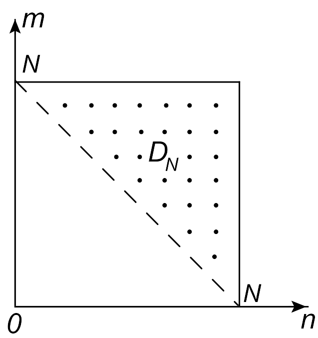
where we used the notation $\delta(true)=1$, $\delta(false)=0$ and the fact that if $m_1+m_2+p=0$ for $|p|\le N$ then $|m_1+m_2|\le N$. Substituting expression for G_{nm} and providing computations we get:

$$
\begin{aligned}
A_N^{(2)}(t_1,t_2) = -\frac{1}{(2\pi)^2}\sum_{\substack{n,m=-N\\ n,m\neq 0}}^{N}\frac{\delta(|n+m|\le N)}{nm}\left(e^{in(t_1-\pi)}-1\right)\left(e^{im(t_2-\pi)}-1\right)+ \\
+\frac{1}{(2\pi)^2}\sum_{\substack{n,m=-N\\ n,m\neq 0}}^{N}\frac{e^{in(t_1-\pi)+im(t_2-\pi)}}{nm}
\end{aligned}
\tag{69}
$$

It's difficult to study the limit $N\to\infty$ in the complete expression (69) for $A^{(2)}$, but we can take a roundabout and compute $A^{(2)}(0,0)$ and $\int A^{(2)}(t_1,t_2)dt_1dt_2$ rather easily. Then we'll check the full answer numerically.

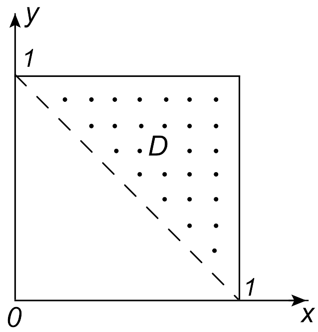
$$
\begin{aligned}
\int A_N^{(2)}(t_1,t_2)dt_1dt_2 &= -\sum_{\substack{n,m=-N\\ n,m\neq 0}}^{N}\frac{\delta(|n+m|\le N)}{nm} = -\sum_{\substack{n,m=-N\\ n,m\neq 0}}^{N}\left[\frac{1}{nm}-\frac{\delta(|n+m|>N)}{nm}\right] \\
&= \sum_{\substack{n,m=-N\\ n,m\neq 0}}^{N}\frac{\delta(|n+m|>N)}{nm} = 2\sum_{(n,m)\in D_N}\frac{1}{nm}
\end{aligned}
\tag{70}
$$

Where D_N is a region illustrated below:



$$
\lim_{N\to\infty} 2\sum_{(n,m)\in D_N}\frac{1}{nm} = \lim_{N\to\infty} 2\sum_{(p,q)\in D_N}\frac{1}{\frac{p}{N}\frac{q}{N}}\frac{1}{N}\frac{1}{N} = 2\int_{(x,y)\in D}\frac{dxdy}{xy} \tag{71}
$$

Where we we've replaced an integral sum by the appropriative integral and the region D is:



Then we can find:

$$\lim_{N\to\infty}\int A_N^{(2)}(t_1,t_2)dt_1dt_2 = 2\int_0^1 \frac{dx}{x}\int_{1-x}^1 \frac{dy}{y} = -2\int_0^1 \frac{dx}{x}\ln(1-x) = \frac{\pi^2}{3} \tag{72}$$

We can also find:

$$A_N^{(2)}(0,0) = \frac{1}{2\pi^2}\sum_{\substack{(n,m)\in D_N\\ n,m\ -\ odd}} \frac{4}{nm} = \frac{1}{2\pi^2}\sum_{\substack{(n,m)\in D_N\\ n,m\ -\ odd}} \frac{1}{\frac{n}{N}\frac{m}{N}}\frac{2}{N}\frac{2}{N} \tag{73}$$

From where we see:

$$\lim_{N\to\infty} A_N^{(2)}(0,0) = \frac{1}{2\pi^2}\int_{(x,y)\in D}\frac{dxdy}{xy} = \frac{1}{12} \tag{74}$$

Comparing (74) and (72) and noticing that $\frac{(2\pi)^2}{12} = \frac{\pi^2}{3}$ one could suppose that $A(t_1,t_2)$ is equal to $\frac{1}{12}$ everywhere except of $t_1 = \pi$ or $t_2 = \pi$ where it vanishes. Such prediction has been checked numerically and turns out to be correct. So we conclude that:

$$A^{(2)}(t_1,t_2) = \begin{cases} \frac{1}{12}, & if\ t_1 \neq \pi \text{ and } t_2 \neq \pi; \\ 0, & otherwise. \end{cases} \tag{75}$$

What is interesting is that the obtained value lies between the two naive predictions: $0 < \frac{1}{12} < \frac{1}{4}$.

3.4 Conclusion to the section

In this section we provided illustrative examples of path integral approach for the phase space. The experience obtained is that all the quantities are well-defined at the tree level, i.e. up to the first order in $\hbar$. At this point the answer happens to be background-independent and the Poisson structure is really restored as expected. However the situation becomes more complicated if we take the loops into account. Naive arguments don't make sense any more - we have to work carefully with respect to the regularization.
We can also notice that (63) deforms the Moyal star-product:

$$\begin{aligned}\langle f_1(\phi(t_1))f_2(\phi(t_2))\rangle =\ & {}^0\langle f_1(\phi(t_1))f_2(\phi(t_2))\rangle + \\ & + \frac{\hbar^2}{24}\theta_{(0)}^{ai_1}\theta_{(0)}^{bi_2}\theta_{(0)}^{ij}\partial_{i_1}\partial_{i_2}(de(x))_{ij}\partial_a f_1(x)\partial_b f_2(x) + o(e) + o(\hbar^2)\end{aligned} \tag{76}$$

which is obviously non-invariant under diffeomorphisms and indicates that there is some kind of anomaly present here – that is the topic of the next section.

4 Anomaly chasing

In this section we are going to analyze our theory behavior with respect to the target-space *nonlinear* diffeomorphisms. We will show that it is not invariant due to the UV cut-off being non-invariant under such diffeomorphisms.

For the purpose of simplicity we will put $x^i = 0$ when providing concrete calculations starting from the subsection 4.2, i.e. all the fields will vanish at $t = \pi$. We shall also assume throughout this section that $e_i(\varphi)$ is a real analytical function whose power series starts from the quadratic term. The same assumption will be about $v^i(\varphi)$ (the vector field describing our infinitesimal diffeomorphism - to be introduced soon).

There are two ways to act with diffeomorphism in a theory with regularization: we either provide naive continuous diffeomorphism and then regularization or provide regularization first and then make diffeomorphism in the finite-dimensional space of regularized fields; this finite-dimensional diffeomorphism imitates the real one in the $N \to \infty$ limit. We'll refer to the first prescription as to the naive one and to the second - as to the proper one.

4.1 Naive prescription

We provide a classical diffeomorphism first:

$$\phi^i = \varphi^i + \epsilon v^i(\varphi) \tag{77}$$

$$\psi^i = \widetilde{\psi}^i + \epsilon \partial_j v^i(\varphi)\widetilde{\psi}^j \tag{78}$$

and study corresponding Ward identities.

We make the following naive assumptions:

1) Supersymmetric measure is invariant - a rather natural assumption.

2) Delta-function regulator is invariant. This should be explained. The delta-function transforms as follows:

$$\begin{aligned}\delta\left(\phi(\pi) - x\right) &= \delta\left(\varphi(\pi) + \epsilon v(\varphi(\pi)) - y - \epsilon v(y)\right) = \\ &= \frac{\delta\left(\varphi(\pi) - y\right)}{|det\left[\delta^i_j + \epsilon\partial_j v^i\left(\varphi(\pi)\right)\right]|} = \frac{\delta\left(\varphi(\pi) - y\right)}{|det\left[\delta^i_j + \epsilon\partial_j v^i\left(y\right)\right]|}\end{aligned} \tag{79}$$

where $x^i = y^i + \epsilon v^i(y)$. We see that the $\frac{1}{|\det(\ldots)|}$ multiplier is naively constant as delta-function guarantees $\varphi^i(\pi) = y^i$, hence it is canceled out from the correlators and we can think of the delta-function as of an invariant object at this point[3].

3) The action transforms in a classical way:

$$S = \int \left\{ \frac{1}{2}\omega^{(0)}_{ij}\varphi^i\dot{\varphi}^j + e_i(\varphi)\dot{\varphi}^i + \epsilon\omega^{(0)}_{ij}v^i(\varphi)\dot{\varphi}^j + \epsilon(\mathcal{L}_v e)_i\dot{\varphi}^i \right\} dt + S_f \tag{80}$$

[3]However when we provide a proper approach, we'll see that the last equality in (79) will not be true. The case is that delta-function non-invariance $\frac{1}{|det\left[\delta^i_j + \epsilon\partial_j v^i(\phi(\pi))\right]|}$ under correlators will give rise to expressions like $\langle\varphi(t)\varphi(\pi)\rangle\langle\varphi(t')\varphi(\pi)\rangle\delta(\varphi(\pi))$ that are *naively* zero but in fact do not vanish as we have seen in the "Anomalous vertex" subsection – because propagator contains jump at $t = \pi$.

Where the fermionic part of the action:

$$S_f = \int \left\{ \omega^{(0)}_{ij} \widetilde{\psi}^i \widetilde{\psi}^j + (de)_{ij} \widetilde{\psi}^i \widetilde{\psi}^j + 2\epsilon \omega^{(0)}_{kj} \partial_i v^k \widetilde{\psi}^i \widetilde{\psi}^j + \epsilon (\mathcal{L}_v de)_{ij} \widetilde{\psi}^i \widetilde{\psi}^j \right\} \tag{81}$$

where $\mathcal{L}_v$ is a Lie derivative with respect to the vector field v along the diffeomorphism.
4) The inserted observables also transform classically: $F[\phi] \to F[\varphi] + \epsilon \int \frac{\delta F[\varphi]}{\delta \varphi^i} v^i$.

At the end of the day we provide mode-regularization, namely use (15). After all these procedures have been done we extract the first order part in ϵ and assume it to vanish (if the invariance is not broken) – that is the way we get the Ward identities. If the identity does not hold – we get an anomaly.

4.1.1 Ward identities and their breakdown

We start with a general interacting theory and are interested in a two-point correlation function:

$$\langle \phi^a(t_1)\phi^b(t_2)\rangle = \frac{\int \mathcal{D}\phi \mathcal{D}\psi \phi^a(t_1)\phi^b(t_2) e^{\frac{i}{\hbar}\int\left\{\frac{1}{2}\omega^{(0)}_{ij}\phi^i\dot{\phi}^j + e_i(\phi)\dot{\phi}^i + ferm.\right\}} \delta\left(\phi^i(\pi)\right)}{\int \mathcal{D}\phi \mathcal{D}\psi e^{\frac{i}{\hbar}\int\left\{\frac{1}{2}\omega^{(0)}_{ij}\phi^i\dot{\phi}^j + e_i(\phi)\dot{\phi}^i + ferm.\right\}} \delta\left(\phi^i(\pi)\right)} \tag{82}$$

and provide the classical diffeomorphism (77)-(78). After that the action transforms into (80), the measure remains the same, as well as the delta-function (as explained above). Observables transform as follows:

$$\phi^a(t_1)\phi^b(t_2) = \varphi^a(t_1)\varphi^b(t_2) + \epsilon\varphi^a(t_1) v^b\left(\varphi(t_2)\right) + \epsilon v^a\left(\varphi(t_1)\right)\varphi^b(t_2) + o(\epsilon) \tag{83}$$

Then we extract the first order in ϵ and assume it to vanish as long as we naively await that coordinate change does not affect the result. Namely the following:

$$\frac{\left\langle \varphi^a(t_1)\varphi^b(t_2)\left(1 + \epsilon\frac{i}{\hbar}\int\left\{\omega^{(0)}_{ij} v^i \dot{\varphi}^j + (\mathcal{L}_v e)_i \dot{\varphi}^i + fermions\right\} dt\right) + \epsilon\left(\varphi^a v^b + v^a \varphi^b\right)\right\rangle}{1 + \epsilon\left\langle \frac{i}{\hbar}\int\left\{\omega^{(0)}_{ij} v^i \dot{\varphi}^j + (\mathcal{L}_v e)_i \dot{\varphi}^i + fermions\right\} dt\right\rangle} \tag{84}$$

should coincide with $\langle \phi^a(t_1)\phi^b(t_2)\rangle$. That is the way to get the Ward identity:

$$\begin{aligned}&\left\langle \varphi^a(t_1) v^b\left(\varphi(t_2)\right)\right\rangle + \left\langle v^a\left(\varphi(t_1)\right)\varphi^b(t_2)\right\rangle + \\ &+\left\langle \varphi^a(t_1)\varphi^b(t_2)\frac{i}{\hbar}\int\left\{\omega^{(0)}_{ij} v^i \dot{\varphi}^j + (\mathcal{L}_v e)_i \dot{\varphi}^i + fermions\right\} dt\right\rangle_{\widehat{ab}} \overset{naively}{=} 0\end{aligned} \tag{85}$$

where the lower index "$\widehat{ab}$" in the last term means that we should not contract $\phi^a(t_1)$ with $\phi^b(t_2)$ - the appropriate term is canceled out by the denominator in (84).

All the propagators of our theory are defined in a perturbative manner. That means that (85) is not just a single identity - it encodes the whole series of identities written in terms of the correlators of quadratic theory $\langle ...\rangle_0$: the zeroth order (in $e(\varphi)$) identity, the first order identity and so on. We will need only the zeroth order one for our purposes. Let us extract it explicitly:

$$^0\langle \varphi^a v^b\rangle + {}^0\langle v^a \varphi^b\rangle + \frac{i}{\hbar}\int {}^0\left\langle \varphi^a(t_1)\varphi^b(t_2)\left\{\omega^{(0)}_{ij} v^i\big(\varphi(t)\big)\dot{\varphi}^j(t) + \right.\right.$$

$$+2\omega_{kj}^{(0)}\partial_i v^k\big(\varphi(t)\big)\widetilde{\psi}^i(t)\widetilde{\psi}^j(t)\Big\}\Big\rangle_{\widehat{ab}} dt \overset{naively}{=} 0 \tag{86}$$

where index "$\widehat{ab}$" again disallows contraction between φ^a and φ^b. We will examine this identity below.

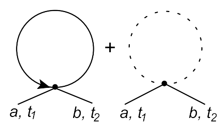
4.1.2 The zeroth order Ward identity

We already stated in the introduction to the chapter that we were interested in nonlinear transformations and that $v^i(\varphi)$ (being thought of as a real analytical function) was a power series starting with quadratic terms: $v^i(\varphi) = \frac{1}{2}\partial_a\partial_b v^i(0)\varphi^a\varphi^b + ...$ Analogously we've stated that $e_i(\varphi) = \frac{1}{2}\partial_a\partial_b e_i(0)\varphi^a\varphi^b + ...$ Taking these into account we conclude that the first two terms in (86) vanish as long as there is either odd number of fields inside the correlator or self-contractions of v^i (which do vanish) present. So we are left with the third term.

If we compute it explicitly we will get an answer that will be a formal power series in $\hbar$. Then vanishing of it means that every coefficient of this formal series vanishes. It's easy to see that this series starts with the $O(\hbar^2)$ - we will calculate only such term. One can see (by observing that there should be 6 fields inside the correlator) that $O(\hbar^2)$ term comes from the cubic part of $v^i(\varphi)$. The only possible contraction in such term (that do not contain self-contractions of v^i which vanish) looks as follows:

$$\frac{i}{\hbar}\int {}^0\left\langle\varphi^a(t_1)\varphi^l(t)\right\rangle\ {}^0\left\langle\varphi^b(t_2)\varphi^s(t)\right\rangle \Big\{\omega_{ij}^{(0)}\partial_l\partial_s\partial_k v^i(0)\ {}^0\langle\varphi^k(t)\dot{\varphi}^j(t)\rangle+$$
$$+2\omega_{kj}^{(0)}\partial_l\partial_s\partial_i v^k(0)\ {}^0\langle\widetilde{\psi}^i(t)\widetilde{\psi}^j(t)\rangle\Big\} dt \tag{87}$$

But it is easy to notice that this expression describes nothing but the two-legs case with self-contraction considered in the "Anomalous vertex" subsection, namely it corresponds to the diagram:



So to write an answer for (87) one may just use the result (63) and make the substitution $(de)_{ij} \to d(\omega_{kp}^{(0)} v^k d\varphi^p)_{ij} = \omega_{kj}^{(0)}\partial_i v^k - \omega_{ki}^{(0)}\partial_j v^k$ in it. So we get:

$$\frac{\hbar^2}{2}\theta_{(0)}^{ai_1}\theta_{(0)}^{bi_2}\theta_{(0)}^{ij}\partial_{i_1}\partial_{i_2}\left(\omega_{kj}^{(0)}\partial_i v^k(0) - \omega_{ki}^{(0)}\partial_j v^k(0)\right)A^{(2)}(t_1,t_2)$$
$$= -\hbar^2\theta_{(0)}^{ai_1}\theta_{(0)}^{bi_2}\partial_{i_1}\partial_{i_2}\partial_i v^i(0)A^{(2)}(t_1,t_2) \tag{88}$$

where we used $\omega_{ij}^{(0)}\theta_{(0)}^{jk} = \delta_i^k$. But we know that $A^{(2)}(t_1,t_2) \neq 0$. So

$$-\hbar^2\theta_{(0)}^{ai_1}\theta_{(0)}^{bi_2}\partial_{i_1}\partial_{i_2}\partial_i v^i(0)A^{(2)}(t_1,t_2) \neq 0 \tag{89}$$

and finally we see that the naive Ward identity is not valid.

4.2 Proper prescription

However we could act differently. We could go to the regularized fields (15) first and then provide diffeomorphism in terms of their Fourier modes. If we put naively:

$$\phi_N^i(t) = \varphi_N^i(t) + \epsilon v^i(\varphi_N(t)) \tag{90}$$

then we'll see that ϕ_N and φ_N connected in such a way can't have an equal support in the Fourier space. Indeed, if the support of φ_N^i is $\{-N, ..., N\}$ then ϕ_N^i will necessarily contain modes higher than N for the case of general diffeomorphism. Only linear transformations do not generate higher modes but if we are interested in a non-linear transformation (and in fact we are) we will always get these higher modes.

The most straightforward way to solve this problem is to throw away these higher modes and to say that the $\{-N, ..., N\}$ part generates the regularized diffeomorphism. We can write down this as follows: $\phi_N^i = \varphi_N^i + \epsilon \left[v^i(\varphi_N)\right]_N$ where the square brackets operation was introduced in the 3.1.2 subsection - it projects on the $span\{e^{-iNt}, ..., e^{iNt}\}$ subspace. Or in terms of (40):

$$\left[v^i\left(\varphi_N(t)\right)\right]_N \equiv \int v^i\left(\varphi_N(t')\right)\delta_N(t'-t)dt' \tag{91}$$

So we provide the diffeomorphism of the mode-regularized theory:

$$\phi_N^i = \varphi_N^i + \epsilon \left[v^i(\varphi_N)\right]_N \tag{92}$$

$$\psi_N^i = \widetilde{\psi}_N^i + \epsilon \left[\partial_j v^i(\varphi_N)\widetilde{\psi}_N^j\right]_N \tag{93}$$

Which is really a diffeomorphism if ϵ is small enough (because it is an identical diffeomorphism for $\epsilon = 0$). As long as this is nothing but the coordinate change in a finite-dimensional integral, the answer will stay unchanged.

The measure will obviously stay invariant under such transformation. But the inserted observables as well as the action and the delta-function will not transform as in the continuous (infinite-dimensional) case as we'll see later - their transformation will give rise to some additional anomalous terms that will explain the anomalies obtained in the Naive approach.

So we start again from the two-point correlator of the interacting theory (82), but written in a mode regularization with cut-off parameter N:

$$\langle\phi_N^a(t_1)\phi_N^b(t_2)\rangle_N = \frac{\int\prod_n d\phi_n d\psi_n \phi_N^a(t_1)\phi_N^b(t_2) e^{\frac{i}{\hbar}\int\left\{\frac{1}{2}\omega_{ij}^{(0)}\phi_N^i\dot{\phi}_N^j + e_i(\phi_N)\dot{\phi}_N^i + ferm.\right\}}\delta\left(\phi_N^i(\pi)\right)}{\int\prod_n d\phi_n d\psi_n e^{\frac{i}{\hbar}\int\left\{\frac{1}{2}\omega_{ij}^{(0)}\phi_N^i\dot{\phi}_N^j + e_i(\phi_N)\dot{\phi}_N^i + ferm.\right\}}\delta\left(\phi_N^i(\pi)\right)} \tag{94}$$

And provide diffeomorphism (92)-(93). The finite-dimensional measure $d\phi d\psi$ stays invariant while the delta-function, the action and the observables do transform nontrivially.

We'll pick up the first order in ϵ below and will study the corresponding Ward identity that will definitely hold. We'll write it in a form "Naive Ward identity = anomalous terms", appropriate for the comparison with our naive Ward identity and will localize the anomaly in such a way. However, we will not analyze all the three non-invariance contributions mentioned above - we'll be satisfied by finding the non-zero effect in the lowest order in perturbations that will be the effect of only one of them .

4.2.1 Delta-function non-invariance

The delta-function transforms as follows:

$$\delta^{(d)}\Big(\phi_N(\pi)\Big) = \delta^{(d)}\Big(\varphi_N(\pi) + \epsilon\left[v\left(\varphi_N(\pi)\right)\right]_N\Big)$$
$$= \delta^{(d)}\Big(\varphi_N(\pi)\Big) +$$
$$+\epsilon\sum_{i=1}^{d}\delta\Big(\varphi_N^1(\pi)\Big)\delta\Big(\varphi_N^2(\pi)\Big)...\delta'\Big(\varphi_N^i(\pi)\Big)...\delta\Big(\varphi_N^d(\pi)\Big)\Big[v^i\left(\varphi_N(\pi)\right)\Big]_N + o(\epsilon) \tag{95}$$

where the expansion in ϵ makes sense only inside the integrals and presence of the delta-functions derivative means that we should integrate by parts[4].

We substitute (95) into the r.h.s of (94), integrate by parts by φ_0^i in the terms containing $\delta'\left(\phi_N^i(\pi)\right)$, extract the $O(\epsilon)$ contribution and get:

$$-\epsilon\left\langle \varphi_N^a(t_1)\varphi_N^b(t_2)\frac{\partial}{\partial\varphi_0^i}\left[v^i\left(\varphi_N(\pi)\right)\right]_N\right\rangle_{\widehat{ab}} - \epsilon\left\langle\left[v^a\left(\varphi_N(\pi)\right)\right]_N\varphi_N^b(t_2)\right\rangle - \epsilon\left\langle\varphi_N^a(t_1)\left[v^b\left(\varphi_N(\pi)\right)\right]_N\right\rangle -$$
$$-\epsilon\left\langle\varphi_N^a(t_1)\varphi_N^b(t_2)\left[v^k\left(\varphi_N(\pi)\right)\right]_N\frac{i}{\hbar}\int\Big\{\partial_k e_i(\varphi_N)\dot\varphi_N^i + \partial_k(de)_{ij}\widetilde{\psi}_N^i\widetilde{\psi}_N^j\Big\}dt\right\rangle_{\widehat{ab}} \tag{97}$$

where the first term arose from the factor $\left[v^i\left(\varphi_N(\pi)\right)\right]_N$ in (95) after integration by parts, the second and the third came from the observables and the fourth term appeared due to the action dependence on φ_0^i. We used the notation $\widehat{ab}$ again to disallow contractions between $\varphi_N^a(t_1)$ and $\varphi_N^b(t_2)$ because the appropriate terms are canceled out due to the denominator of (94). We can also notice that $\frac{\partial}{\partial\varphi_0^i}\left[v^i\left(\varphi_N(\pi)\right)\right]_N = \left[\partial_i v^i\left(\varphi_N(\pi)\right)\right]_N$. It is convenient to rewrite (97) in a following compact way:

$$-\epsilon\left\langle D\{\phi_N^a(t_1)\phi_N^b(t_2)\}\right\rangle \tag{98}$$

with D being a differential operator:

$$D = \left[\partial_i v^i\left(\varphi_N(\pi)\right)\right]_N + \left[v^k\left(\varphi_N(\pi)\right)\right]_N\frac{i}{\hbar}\int\Big\{\partial_k e_i(\varphi_N)\dot\varphi_N^i + \partial_k(de)_{ij}\widetilde{\psi}_N^i\widetilde{\psi}_N^j\Big\}dt +$$
$$+\left[v^i\left(\varphi_N(\pi)\right)\right]_N\frac{\partial}{\partial\varphi_0^i} \tag{99}$$

Notice that this really contributes because of the projector $[..]_N$ presence: $\partial_i v^i(\varphi_N(\pi))$ and $v^i(\varphi(\pi))$ vanish inside the correlators (due to $\delta^{(d)}\left(\varphi(\pi)\right)$) while $\left[\partial_i v^i(\varphi_N(\pi))\right]_N$ and $\left[v^i(\varphi(\pi))\right]_N$ don't.

The delta-function non-invariance was absent in the naive approach at all.

4.2.2 Action transform and its non-invariance

After we provided the diffeomorphism (92)-(93), the change of the action is:

$$\Delta S_{reg} = \epsilon\int\{\omega_{ij}^{(0)}\left[v^i(\varphi_N)\right]_N\dot\varphi_N^j + \left[v^k(\varphi_N)\right]_N(de(\varphi_N))_{ki}\dot\varphi_N^i + 2\omega_{ij}^{(0)}\Big[\partial_k v^i(\varphi_N)\widetilde{\psi}_N^k\Big]_N\widetilde{\psi}_N^j +$$

[4] In fact, the following holds:

$$\int dx\delta\left(x+\epsilon g(x)\right)f(x) = \int dx\delta(x)f(x) + \epsilon\int dx\delta'(x)g(x)f(x) + o(\epsilon) \tag{96}$$

with appropriate restrictions on $g(x)$. In (95) we used this "Taylor expansion" for the delta-function.

$$+\left[v^k\right]_N \partial_k(de(\varphi_N))_{ij}\widetilde{\psi}^i_N\widetilde{\psi}^j_N + 2(de(\varphi_N))_{ij}\left[\partial_k v^i(\varphi_N)\widetilde{\psi}^k\right]_N \widetilde{\psi}^j\}dt \tag{100}$$

Notice that $\int\left[v^i(\varphi_N(t))\right]_N \dot{\varphi}^j_N(t)dt = \int\int v^i(\varphi_N(t'))\delta_N(t'-t)dt'\dot{\varphi}^j_N(t)dt = \int v^i(\varphi_N(t'))\dot{\varphi}^j_N(t')dt'$ and analogously $\int\left[\partial_k v^i(\varphi_N)\widetilde{\psi}^k_N\right]_N \widetilde{\psi}^j_N dt = \int \partial_k v^i(\varphi_N)\widetilde{\psi}^k_N\widetilde{\psi}^j_N dt$. So we have:

$$\Delta S_{reg} = \epsilon\int\{\omega^{(0)}_{ij}v^i(\varphi_N)\dot{\varphi}^j_N + \left[v^k(\varphi_N)\right]_N (de(\varphi_N))_{ki}\dot{\varphi}^i_N + 2\omega^{(0)}_{ij}\partial_k v^i(\varphi_N)\widetilde{\psi}^k_N\widetilde{\psi}^j_N +$$

$$+\left[v^k\right]_N \partial_k(de(\varphi_N))_{ij}\widetilde{\psi}^i_N\widetilde{\psi}^j_N + 2(de(\varphi_N))_{ij}\left[\partial_k v^i(\varphi_N)\widetilde{\psi}^k\right]_N \widetilde{\psi}^j\}dt \tag{101}$$

We see that the last expression is different from the one obtained in the naive case (see (80 – 81)). Namely:

$$A = \frac{\Delta S_{naive} - \Delta S_{reg}}{\epsilon} = \int dt\Big\{\left(v^k(\varphi_N) - \left[v^k(\varphi_N)\right]_N\right)(de(\varphi_N))_{ki}\dot{\varphi}^i_N +$$

$$+\left(v^k - \left[v^k\right]_N\right)\partial_k(de(\varphi_N))_{ij}\widetilde{\psi}^i_N\widetilde{\psi}^j_N +$$
$$+2(de(\varphi_N))_{ij}\left(\partial_k v^i(\varphi_N)\widetilde{\psi}^k - \left[\partial_k v^i(\varphi_N)\widetilde{\psi}^k\right]_N\right)\widetilde{\psi}^j\Big\} \tag{102}$$

This difference will be referred to as the action non-invariance. Notice that it naively vanishes in the $N\to\infty$ limit. However it still can contribute if one takes off the regularization carefully - at the end of the computations.

4.2.3 Observables non-invariance

The observables transform as follows:

$$\phi^a_N(t_1)\phi^b_N(t_2) \to \varphi^a_N(t_1)\varphi^b_N(t_2) + \epsilon\varphi^a_N(t_1)\left[v^b(\varphi_N(t_2))\right]_N + \epsilon\left[v^a(\varphi_N(t_1))\right]_N\varphi^b_N(t_2) \tag{103}$$

Non-invariance generated by this transformation, namely the difference from the naive setup case is:

$$O = \frac{naive - proper}{\epsilon} = \varphi^a_N(t_1)\left(v^b(\varphi_N(t_2)) - \left[v^b(\varphi_N(t_2))\right]_N\right) +$$
$$+\left(v^a(\varphi_N(t_1)) - \left[v^a(\varphi_N(t_1))\right]_N\right)\varphi^b_N(t_2) \tag{104}$$

Again it naively vanishes in the $N\to\infty$ limit.

Our following step will be to gather the $O(\epsilon)$ terms altogether and to obtain the Ward identities. As long as in the proper approach we make diffeomorphism of the finite-dimensional (i.e. well-defined) integral, these identities will really hold and will show why the naive approach failed.

4.2.4 Ward identity

Let us gather together all the terms above and write the $O(\epsilon)$ contribution. It should vanish and certainly will. We'll write it in the form "naive Ward identity (N.W.I.)" = "proper non-invariant terms":

$$N.W.I. = \left\langle D\left\{\varphi^a_N(t_1)\varphi^b_N(t_2)\right\}\right\rangle^{\widehat{ab}}_N + \left\langle A\varphi^a_N(t_1)\varphi^b_N(t_2)\right\rangle^{\widehat{ab}}_N + \langle O\rangle_N \tag{105}$$

where "N.W.I." equals to the l.h.s of (85). Notice that D is a differential operator and we have:

$$\left\langle D\left\{\varphi_N^a(t_1)\varphi_N^b(t_2)\right\}\right\rangle_{\widehat{ab}} = \left\langle \varphi_N^a(t_1)\varphi_N^b(t_2)\left[\partial_i v^i\left(\varphi_N(\pi)\right)\right]_N\right\rangle_{\widehat{ab}} + \\ + \left\langle \left[v^a\left(\varphi_N(\pi)\right)\right]_N \varphi_N^b(t_2)\right\rangle + \left\langle \varphi_N^a(t_1)\left[v^b\left(\varphi_N(\pi)\right)\right]_N\right\rangle + \\ + \left\langle \varphi_N^a(t_1)\varphi_N^b(t_2)\left[v^k\left(\varphi_N(\pi)\right)\right]_N \frac{i}{\hbar}\int\left\{\partial_k e_i(\varphi_N)\dot{\varphi}_N^i + \partial_k(de)_{ij}\widetilde{\psi}_N^i\widetilde{\psi}_N^j\right\}dt\right\rangle_{\widehat{ab}} \tag{106}$$

So we have three possible anomalous terms - D describes delta-function non-invariance, A describes action non-invariance and O describes observables non-invariance.

4.2.5 The zeroth order identity

Similar to the naive setup, we pick up the zeroth order in $e_i(\varphi)$ from (105). Let us analyze different contributions:

1) O from (104) doesn't contribute as

$$\left\langle \varphi_N^a(t_1)\left[v^b(\varphi_N(t_2))\right]_N\right\rangle_N = \int \langle \varphi^a(t_1)v^b(\varphi_N(t))\rangle_N \delta_N(t-t_2)dt$$

and in the zeroth order in $e_i(\varphi)$ we have

$$^0\left\langle \varphi^a(t_1)v^b(\varphi_N(t))\right\rangle_N = 0 \tag{107}$$

as long as $v^b(\varphi)$ starts from the quadratic terms and we have $^0\langle\varphi_N^i(t)\varphi_N^j(t)\rangle_N = 0$ in our prescription. Moreover, in our case O won't contribute in all orders in $e_i(\varphi)$ as long as $\left[v^a\left(\varphi_N(t)\right)\right]_N - v^a\left(\varphi_N(t)\right)$ contains only higher Fourier modes in t - modes with absolute value of mode number greater than N. But this means that every Fourier mode with finite number vanishes in the $N \to \infty$ limit and if the answer exists at all - it should be zero. In general we could consider such observables that this argument wouldn't be true (e.g. integrated observables), but in our case O doesn't contribute.

2) D from (106) contributes but only the first term is relevant. The second and the third terms vanish in the zeroth order for the same reasons as (107). The last term in (106) is of the higher order in $e_i(\varphi)$ and we neglect it.

3) A from (102) is of the higher order in $e_i(\varphi)$ and thus is not relevant.

4) The zeroth order (in $e_i(\varphi)$) part of the l.h.s. of (105), i.e. the zeroth order part of the naive Ward identity (85) has already been calculated in the 4.1.2 subsection and is given by (89).

So after all we are left with the following terms:

$$-\hbar^2\theta_{(0)}^{ai_1}\theta_{(0)}^{bi_2}\partial_{i_1}\partial_{i_2}\partial_i v^i(0)A^{(2)}(t_1,t_2) = \,^0\left\langle \varphi_N^a(t_1)\varphi_N^b(t_2)\left[\partial_i v^i\left(\varphi_N(\pi)\right)\right]_N\right\rangle^{\widehat{ab}} \tag{108}$$

But one can notice that $\left[\partial_i v^i(\varphi(\pi))\right]_N$ is nothing but the anomalous vertex (56) where one has replaced $e_i(\varphi)$ by $\omega_{ki}^{(0)}v^k$. Thus we can use the result (63) for the r.h.s. of (108) and find that (108) really holds - the anomaly of the naive Ward identity is canceled out by the term $\left\langle \varphi_N^a(t_1)\varphi_N^b(t_2)\left[\partial_i v^i\left(\varphi_N(\pi)\right)\right]_N\right\rangle_0^{\widehat{ab}}$ which describes delta-function non-invariance.

This shows that in the lowest order in perturbations the Ward identity holds. Although we are not analyzing the higher orders, it's clear that it holds there too, just because the proper prescription deals with finite-dimensional space of fields.

5 Discussion and conclusions

The current paper was devoted to the study of the general phase space covariance of quantum mechanics. We have been working in the path integral formalism with a special model theory (3). During this study we have seen that the naive bosonic path integral (3) (in the phase space) is an ill-defined object as it gives rise to an extremely divergent theory. The solution of the problem was to “supersymmetrize” it by adding anticommuting ghost fields transforming in a proper way, such that the modified theory became finite and well-defined. At the end of the day we found that the quantum answer was not invariant under classical diffeomorphisms - it behaved anomalously. And the source of the anomaly was the whole path integral becoming non-invariant after regularization.

In the first half of the paper the super-improvement was discussed. The framework was formulated and its self-consistency was proven. The main idea of the method was to replace a naive bosonic measure by a super-modified one:

$$\mathcal{D}\phi \to \mathcal{D}\phi\mathcal{D}\psi \; e^{\frac{i}{\hbar}\int \omega_{ij}(\phi)\psi^i\psi^j dt} \tag{109}$$

Then some illustrative examples as well as an important discussion of the anomalous vertex were performed.

From the “Examples” chapter we have studied that our theory has a peculiar issue connected with the specific behavior of correlators. Correlators of observables in the phase space contain jumps in their time dependences. The value of the given correlation function at the point of this jump is not fixed in general and depends on the regularization (for example it depends on the form of UV cut-off in our case). When providing perturbative analysis one often has to compute the integrals of the form $\int f(t)\delta(t-t_0)dt$, where the function $f(t)$ contains jump at the point $t = t_0$ – we face such troubles when considering $A^{(n)}(t_1, ..., t_n)$. Such integrals need regularization badly and their values depend on it.

Then the problem of general covariance was discussed. We were interested in a diffeomorphism action in our theory. The naive approach to the subject was to make a classical diffeomorphism (77, 78) in our super-space first, not worrying about regularization at all. Such way of thinking gave rise to the anomalous Ward identity (85). In the lowest order in perturbations it was found (see (87)) that the source of the anomaly in such approach was the anomalous vertex had been obtained before. We have seen that, roughly speaking, the ambiguity appearing when integrating jump with the delta-function generated the anomaly in the lowest order.

However then we described a proper prescription, which demonstrated the real origin of non-invariance. The proper prescription said, that one had to introduce the regularization first and then work in terms of the regularized finite-dimensional theory. The notion of diffeomorphism had to be modified as long as the classical diffeomorphisms didn’t respect the mode-regularization - they spoiled the cut-off, generating modes with mode number above the cut-off parameter N. That’s why we had to introduce the “regularized” diffeomorphisms (92, 93) - finite-dimensional diffeomorphisms of the regularized theory imitating classical ones in the $N \to \infty$ limit. However all the components of the path integral construction

became non-invariant after such modification, or more precisely, they transformed in a non-classical, non-covariant way under these regularized diffeomorphisms (for the fixed finite cut-off parameter N). These non-covariances are treated as the real origin of the anomaly in a mode-regularization prescription.

We have computed separately all the potentially anomalous terms - the delta-function anomaly D (see (99)), the action anomaly A (see (102)) and the observables anomaly O (see (104)). All these three terms could in principle contribute if we considered higher orders of perturbation theory and arbitrary observables. However we were satisfied by locating the lowest non-trivial anomalous contribution - it came from the delta-anomaly D in our case. Notice that in general one could replace the delta-function zero-mode regulator (the evaluation observable) by a non-zero Hamiltonian. In such case the delta-function non-invariance would be replaced by the Hamiltonian non-invariance (which would be almost the same as the action non-invariance).

In general it would be interesting and important to study the higher orders (where A and for special observables also O starts to contribute) and to explore how the classical symmetries are deformed at the quantum level and give rise to the Kontsevich's L_∞-morphism. But we leave this beyond the current paper as a topic of an upcoming research.

One could ask an interesting question, whether the claimed anomaly was an artifact of the regularization (which by itself was non-invariant under diffeomorphisms), or it was a fundamental property of the theory. We know that there is no quantum answer for the problem (3) that could be invariant under classical diffeomorphisms. Thus we could conclude that the anomaly was a fundamental property of the theory. And what we have done is just the analysis of this anomaly in a mode-regularization prescription. However it would be much better to study the subject in another framework with some alternative regularization as well in order to understand if anything depends on it. This alternative regularization is time-slicing (or equivalently discretization, or lattice regularization). In such case one replaces continuous time by the set of N points, continuous fields $\phi^i(t)$, $\psi^i(t)$ – by the set of their values at these points: ϕ^i_k, ψ^i_k and considers the following object:

$$\int \prod_{k,i} d\phi^i_k d\psi^i_k \prod_i \delta(\phi^i_N - x^i) \exp \frac{i}{\hbar} \sum_k \left\{ \alpha_i(\phi_k)(\phi^i_{k+1} - \phi^i_k) + \omega_{ij}(\phi_k)\psi^i_k\psi^j_k \right\} \tag{110}$$

where measure, delta-function and $\omega_{ij}(\phi_k)\psi^i_k\psi^j_k$ term are invariant under classical diffeomorphisms! The only term that behaves in a non-classical way is $\alpha_i(\phi_k)(\phi^i_{k+1} - \phi^i_k)$. However despite of this seeming simplification, the time-slicing approach needs some extra analysis and additional assumptions as in our special case the proper continuous limit $N \to \infty$ doesn't exist even for the *quadratic* theory! This is connected with the delta-function making an illegal operation - fixing coordinates and momentums at the same point. So one needs to think over this subtle issue carefully and thus we leave this for an upcoming research as well.

A Bosonic divergences

First we need to find a propagator:

$$^0\langle\phi^i(t_1)\phi^j(t_2)\rangle = \frac{\int\prod_{n,i} d\phi^i_n \prod_{i=1}^d \delta(\sum_n \phi^i_n(-1)^n - x^i)\phi^i(t_1)\phi^j(t_2)e^{-\frac{1}{\hbar}\sum_n \pi\omega^{(0)}_{ij}\phi^i_{-n}n\phi^j_n}}{\int\prod_{n,i} d\phi^i_n \prod_{i=1}^d \delta(\sum_n \phi^i_n(-1)^n - x^i)e^{-\frac{1}{\hbar}\sum_n \pi\omega^{(0)}_{ij}\phi^i_{-n}n\phi^j_n}} \tag{111}$$

$$^0\langle\phi^i(t_1)\phi^j(t_2)\rangle = x^i x^j + i\hbar\theta^{ij}_{(0)}G(t_1,t_2) \tag{112}$$

Where $\theta_{(0)} = \left(\omega^{(0)}\right)^{-1}$, $G(t_1,t_2) = \sum_{n,m} G_{n,m}e^{int_1+imt_2}$ and:

$$G_{n,m} = \frac{1}{2\pi i}\left\{\frac{\delta_{n+m}}{n} - \frac{\delta_m}{n}(-1)^n + \frac{\delta_n}{m}(-1)^m\right\}$$
$$G_{0,0} = 0 \tag{113}$$

Notice that we think of it as of an infinite-dimensional matrix, inverse to $\pi\omega^{(0)}_{ij}n\delta_{n+m}$ - again we don't care about correctness yet.

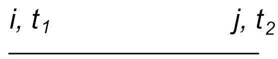
From (6) we see that the only interaction in our theory is $e_i(\phi)\dot\phi^i$ which is in fact a series of interactions: $e_i(\phi)\dot\phi^i = \frac{1}{2}\partial_{a_1}\partial_{a_2}e_i(0)\phi^{a_1}\phi^{a_2}\dot\phi^i + \frac{1}{3!}\partial_{a_1}\partial_{a_2}\partial_{a_3}e_i(0)\phi^{a_1}\phi^{a_2}\phi^{a_3}\dot\phi^i + ...$ (we can neglect linear term due to the periodic boundary conditions in time and get rid of the quadratic term by relating it to the free part of the action).

It makes sense to introduce diagram rules for the further considerations.

Diagram rules in the coordinate space

Let us denote a free propagator by the line:

$$^0\langle\phi^i(t_1)\phi^j(t_2)\rangle = \tag{114}$$



and introduce a convenient notation for the interaction - a vertex with an arrow on the leg containing the time derivative (as in [[article]]):

$$\frac{1}{k!}\partial_{a_1}...\partial_{a_k}e_i(0)\phi^{a_1}(t)...\phi^{a_k}(t)\dot\phi^i(t) = \tag{115}$$

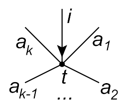


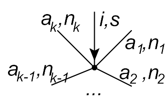
Diagram rules in the momentum space

Again denote a free propagator by the line:

$$^0\langle\phi^i_n\phi^j_m\rangle = \tag{116}$$

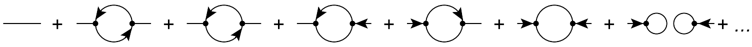
and interaction - by the vertex with an arrow on the leg containing the time derivative:

$$\frac{2\pi}{k!}\partial_{a_1}...\partial_{a_k}e_i(0)\delta_{n_1+n_2+...+n_k+s}\phi^{a_1}_{n_1}...\phi^{a_k}_{n_k}s\phi^i_s = \tag{117}$$



Now let us try to compute an interacting propagator up to the second order, or at least observe, how these computations could be provided. For simplicity assume that $e_i(\phi) =$

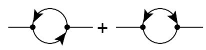
$a_{i,j,k}\phi^j\phi^k$ contains only quadratic terms. Then an interacting propagator is given by the following series of graphs:

$$^0\langle\phi^i(t_1)\phi^j(t_2)e^{\frac{i}{\hbar}\int e}\rangle$$
$$= \quad (118)$$


As we see, they contain several types of loops. These loops bring some divergences and ambiguities. For example the loop:

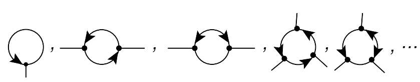
corresponds in a momentum space to $\sum_{n,m} s\phi^i_s\delta_{n+m+s}G_{n,m} = \frac{s\phi^i_s}{2\pi i}\delta_s\sum_n \frac{1}{n}$ which is a logarithmic divergent expression. But here we can notice some effect - in the case of an odd-dimensional space-time (which is our case) logarithmic divergences are not real divergences - they don't need any counterterms to cancel them, all we need to do is to provide some kind of ultroviolet regularization - an UV cut-off in our case - and the logarithmic divergency won't take place any more. Instead of it a logarithmic ambiguity appears - the answer depends on the form of the UV cut-off. This shows us that at least we have to replace a formal infinite sum in (4) by the finite one with some limiting UV parameter N.
The loop of the type:

As well as the loops:



are not logarithmic but linear divergent! Indeed, every propagator is proportional to the inverse power of the momentum and every arrow on the leg (indicating derivative) is "proportional" to the first power of the momentum. Thus as soon as the loop contains equal numbers of propagators and arrows, it can produce a linear divergence. Indeed, the first loop is proportional to $\sum_{n,m}\delta_{n+m+s}nG_{nm} = \frac{\delta_s}{2\pi i}\sum_n 1 - \frac{(-1)^s}{2\pi i}$ which is a linear divergence. The same can be obtained for the second one. These are real divergences that should be cancelled by some manually added counterterms at this point of consideration.

The situation becomes really weird if we notice that there exists an infinite series of wheel-like diagrams all being linearly divegrent:

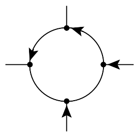


such series often indicates that the theory is non-renormalizable as long as every set of diagrams of the series with the fixed number of external legs needs a new type of counterterm to be introduced. It looks natural and strange at once. People know - interactions containing derivatives often happen to be non-renormalizable, but they are also used to the fact that one-dimensional theories are finite. So what we have to do?

Again we see that we need to provide an ultraviolet cut-off to make a sence of all this. And more - we need to add counterterms. The good thing is that all the necessary counterterms are encoded in (8) which leads us to the super-improvement idea.

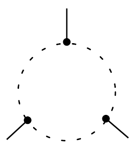
B Cancelation of divergences

Lemma 1: Every non-ghost loop that contains less number of internal arrows (indicating time derivatives) than internal propagators is finite. For example:

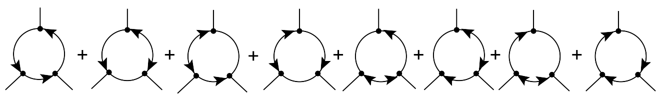


Sketch of proof: Notice that an ultraviolet behavior doesn't depend on the infrared one and thus we can think for a moment that $G_{nm} = \frac{1}{2\pi i}\frac{\delta_{n+m}}{n}$, i.e. that the bosonic propagator conserves momentum (one can check if needed that this really doesn't affect the divergences). After that we can provide the following argument: suppose the momentum p circulates over the loop; when $p \to \infty$ every propagator carries the factor $\frac{1}{p}$ and every internal arrow gives rise to the factor p; then as soon as we have more propagators than arrows, total loop is proportional at least to the first inverse power of p; thus it converges in a sense of mode regularization due to the Observation 4 - because in our theory expressions like $\sum \frac{1}{p}$ are finite.

Now consider a ghost loop with n vertexes:



Lemma 2: this diagram cancels the divergent part of the sum of non-ghost loops with n vertexes and with equal numbers of internal propagators and internal arrows:



Proof: Let us introduce a convenient notation for such diagrams. Go around the diagram in a clockwise direction, numerate vertexes and for the k'th vertex say that it is of the type "c" if the appropriate arrow is in the clockwise direction to the vertex and we say that it is of the type "a" if the appropriate arrow is in the anticlockwise direction to the vertex. We also associate indexes i_k and j_k to each vertex (i - to the internal leg without arrow and j - to the internal leg with arrow). Denote such a loop as follows: $Loop\left[{}^{i_1 j_1}_{d_1}, {}^{i_2 j_2}_{d_2}, \dots, {}^{i_n j_n}_{d_n}\right]$ where d_k indicates the type of the vertex and is either "c" or "a". We'll also associate an analytical expression to the $Loop[\,]$ in a natural way:

$$Loop\left[{}^{i_1 j_1}_{c}, {}^{i_2 j_2}_{c}, \dots, {}^{i_n j_n}_{c}\right] = {}^0\langle \dot{\phi}^{j_1}(t_1)\phi^{i_2}(t_2)\rangle\ {}^0\langle \dot{\phi}^{j_2}(t_2)\phi^{i_3}(t_3)\rangle \dots {}^0\langle \dot{\phi}^{j_n}(t_n)\phi^{i_1}(t_1)\rangle \qquad (119)$$

which is nothing but the product of internal propagators along the loop. In terms of $Loop[\,]$ we can write the total contribution of our loops as follows (we don't mind about external

legs explicitly):

$$ {}^0\left\langle \ldots \left(\frac{i}{\hbar}\right)^n \int dt_1\ldots dt_n \sum_{d_k\in\{c,\ a\}} \partial_{i_1} e_{j_1}\left(\phi(t_1)\right)\ldots\partial_{i_n} e_{j_n}\left(\phi(t_n)\right) Loop\left[{}^{i_1 j_1}_{d_1},{}^{i_2 j_2}_{d_2},\ldots,{}^{i_n j_n}_{d_n}\right]\ldots\right\rangle \tag{120} $$

Notice that:

$$ \int dt_1 \partial_{i_1} e_{j_1} Loop\left[{}^{i_1 j_1}_{a},{}^{i_2 j_2}_{c},\ldots,{}^{i_n j_n}_{c}\right] $$

$$ = \int dt_1 \partial_{i_1} e_{j_1}\ {}^0\langle\phi^{i_1}(t_1)\phi^{i_2}(t_2)\rangle\ {}^0\langle\dot\phi^{j_2}(t_2)\phi^{i_3}(t_3)\rangle\ldots{}^0\langle\dot\phi^{j_n}(t_n)\dot\phi^{j_1}(t_1)\rangle = \text{(intagrate by parts)} $$

$$ = -\int dt_1 \partial_{i_1} e_{j_1}\ {}^0\langle\dot\phi^{i_1}(t_1)\phi^{i_2}(t_2)\rangle\ {}^0\langle\dot\phi^{j_2}(t_2)\phi^{i_3}(t_3)\rangle\ldots{}^0\langle\dot\phi^{j_n}(t_n)\phi^{j_1}(t_1)\rangle - $$

$$ -\int dt_1 \dot\phi^k(t_1)\partial_p\partial_{i_1} e_{j_1}\ {}^0\langle\phi^{i_1}(t_1)\phi^{i_2}(t_2)\rangle\ {}^0\langle\dot\phi^{j_2}(t_2)\phi^{i_3}(t_3)\rangle\ldots{}^0\langle\dot\phi^{j_n}(t_n)\phi^{j_1}(t_1)\rangle = (i_1\leftrightarrow j_1) $$

$$ = -\int dt_1 \partial_{j_1} e_{i_1} Loop\left[{}^{i_1 j_1}_{c},{}^{i_2 j_2}_{c},\ldots,{}^{i_n j_n}_{c}\right] + \text{regular part} \tag{121} $$

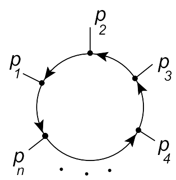
Now we can see that:

$$ \begin{aligned} &\int dt_1 \partial_{i_1} e_{j_1}\left(Loop\left[{}^{i_1 j_1}_{a},{}^{i_2 j_2}_{c},\ldots,{}^{i_n j_n}_{c}\right] + Loop\left[{}^{i_1 j_1}_{c},{}^{i_2 j_2}_{c},\ldots,{}^{i_n j_n}_{c}\right]\right) \\ &= \int dt_1 \left(\partial_{i_1} e_{j_1} - \partial_{j_1} e_{i_1}\right) Loop\left[{}^{i_1 j_1}_{c},{}^{i_2 j_2}_{c},\ldots,{}^{i_n j_n}_{c}\right] + \text{regular part} \\ &= \int dt_1 de_{i_1 j_1} Loop\left[{}^{i_1 j_1}_{c},{}^{i_2 j_2}_{c},\ldots,{}^{i_n j_n}_{c}\right] + \text{regular part} \end{aligned} \tag{122} $$

Analogously one can show that:

$$ (120) = {}^0\left\langle \ldots \left(\frac{i}{\hbar}\right)^n \int dt_1\ldots dt_n (de)_{i_1 j_1}\left(\phi(t_1)\right)\ldots(de)_{i_n j_n}\left(\phi(t_n)\right) Loop\left[{}^{i_1 j_1}_{c},\ldots,{}^{i_n j_n}_{c}\right]\ldots\right\rangle + \text{reg} \tag{123} $$

Naively one can see that $Loop\left[{}^{i_1 j_1}_{c},\ldots,{}^{i_n j_n}_{c}\right] \sim \delta(0) + reg$. Let us calculate it in a mode-regularization framework. Consider a loop in the momentum space:



To calculate its divergent part we can again put $G_{n,m} = \frac{1}{2\pi i}\frac{\delta_{n+m}}{n}$, $G_{0,0} = 0$ and also vanish external momentums $p_k = 0$. Then we can put ${}^0\langle(\dot\phi^i)_n\phi^j_m\rangle = i\hbar\theta^{ij}_{(0)}\frac{\delta_{n+m}}{2\pi}$ and finally get an expression for the loop:

$$ \left(\frac{i\hbar}{2\pi}\right)^n \theta^{j_1 i_2}_{(0)}\theta^{j_2 i_3}_{(0)}\ldots\theta^{j_n i_1}_{(0)} \sum_{\{q\},\{m\}} \delta_{q_1+m_2}\delta_{q_2+m_3}\ldots\delta_{q_n+m_1}\delta_{q_1+m_1}\delta_{q_2+m_2}\ldots\delta_{q_n+m_n} \tag{124} $$

Here $\delta_{q_1+m_1},\ \delta_{q_2+m_2},\ldots,\delta_{q_n+m_n}$ come from the vertexes. Providing trivial summation one gets:

$$ (124) = \left(\frac{i\hbar}{2\pi}\right)^n \theta^{j_1 i_2}_{(0)}\theta^{j_2 i_3}_{(0)}\ldots\theta^{j_n i_1}_{(0)} \sum_{\substack{q=-N..N\\ q\neq 0}} 1 = \left(\frac{i\hbar}{2\pi}\right)^n \theta^{j_1 i_2}_{(0)}\theta^{j_2 i_3}_{(0)}\ldots\theta^{j_n i_1}_{(0)} 2N \tag{125} $$

that is a linear divergent expression. Now let us consider a ghost loop. It's contribution:

$$
{}^{0}\Big\langle \dots \left(\frac{i}{\hbar}\right)^{n} \int dt_1 \dots dt_n de_{i_1 j_1}(\phi(t_1)) \dots de_{i_n j_n}(\phi(t_n)) (-1)^{2n-1} 2^n \times \\
\times \left[{}^{0}\langle \psi^{j_1}\psi^{i_2}\rangle {}^{0}\langle \psi^{j_2}\psi^{i_3}\rangle \dots {}^{0}\langle \psi^{j_n}\psi^{i_1}\rangle\right] \dots \Big\rangle \qquad (126)
$$

where factor $(-1)^{2n-1} = -1$ reflects the parity of permutation we have to perform to transform $\psi^{i_1}\psi^{j_1}\psi^{i_2}\psi^{j_2}...\psi^{i_n}\psi^{j_n}$ into $\psi^{j_1}\psi^{i_2}\psi^{j_2}\psi^{i_3}...\psi^{j_n}\psi^{i_1}$ and factor 2^n reflects an additional symmetry of the diagram: we can change $\psi^{i_k} \leftrightarrow \psi^{j_k}$ as they contribute into the vertex in a similar way. Divergent part of (126) is proportional to $\delta(0)$, now we want to show that it is exactly the divergence of (120). To do that we calculate the ghost loop in the momentum space (using ghost propagator):

$$
-2^n \left(\frac{i\hbar}{4\pi}\right)^n \theta_{(0)}^{j_1 i_2}\theta_{(0)}^{j_2 i_3}\dots\theta_{(0)}^{j_n i_1} \sum_{\{q\},\{m\}} \delta_{q_1+m_2}\delta_{q_2+m_3}\dots\delta_{q_n+m_1}\delta_{q_1+m_1}\delta_{q_2+m_2}\dots\delta_{q_n+m_n} \qquad (127)
$$

Here again $\delta_{q_1+m_1},\ \delta_{q_2+m_2},\dots,\delta_{q_n+m_n}$ come from the vertexes. After trivial summation:

$$
(127) = -\left(\frac{i\hbar}{2\pi}\right)^n \theta_{(0)}^{j_1 i_2}\theta_{(0)}^{j_2 i_3}\dots\theta_{(0)}^{j_n i_1} \sum_{q=-N..N} 1 = -\left(\frac{i\hbar}{2\pi}\right)^n \theta_{(0)}^{j_1 i_2}\theta_{(0)}^{j_2 i_3}\dots\theta_{(0)}^{j_n i_1}(2N+1) \qquad (128)
$$

Now we see that (128) and (125) differ by sign and also by the finite contribution of ghost's zero mode. Then when (120) and (126) are taken together, divergent parts do cancel each other and finally we get the statement of the Lemma 2.

Observation 5: Now when we have Lemma 1 and Lemma 2 in our disposal, we automatically have the fact that any correlation function in momentum space $\langle\phi_{m_1}^{i_1}...\phi_{m_k}^{i_k}\rangle$ is finite. Indeed, application of Lemma 2 gives that we can get rid of ghosts at all together with non-ghost loops with equal numbers of propagators and internal arrows (note that number of internal arrows is less or equal to the number of propagators). After all Lemma 1 guarantees that all that is left is finite.

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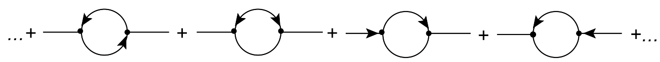
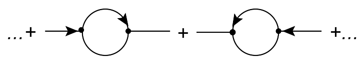
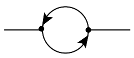
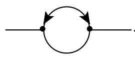




…+  +  +…